

PowerPoint to accompany

# **Introduction to MATLAB for Engineers, Third Edition**

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## **Chapter 8 Linear Algebraic Equations**



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A *singular* problem refers to a set of equations having either no unique solution or no solution at all. For example, the set

$$3x - 4y = 5$$

$$6x - 8y = 10$$

is singular and has no unique solution because the second equation is identical to the first equation, multiplied by 2. The graphs of these two equations are identical. All we can say is that the solution must satisfy  $y = (3x - 5)/4$ , which describes an infinite number of solutions.

The equations

$$6x - 10y = 2$$

$$3x - 4y = 5$$

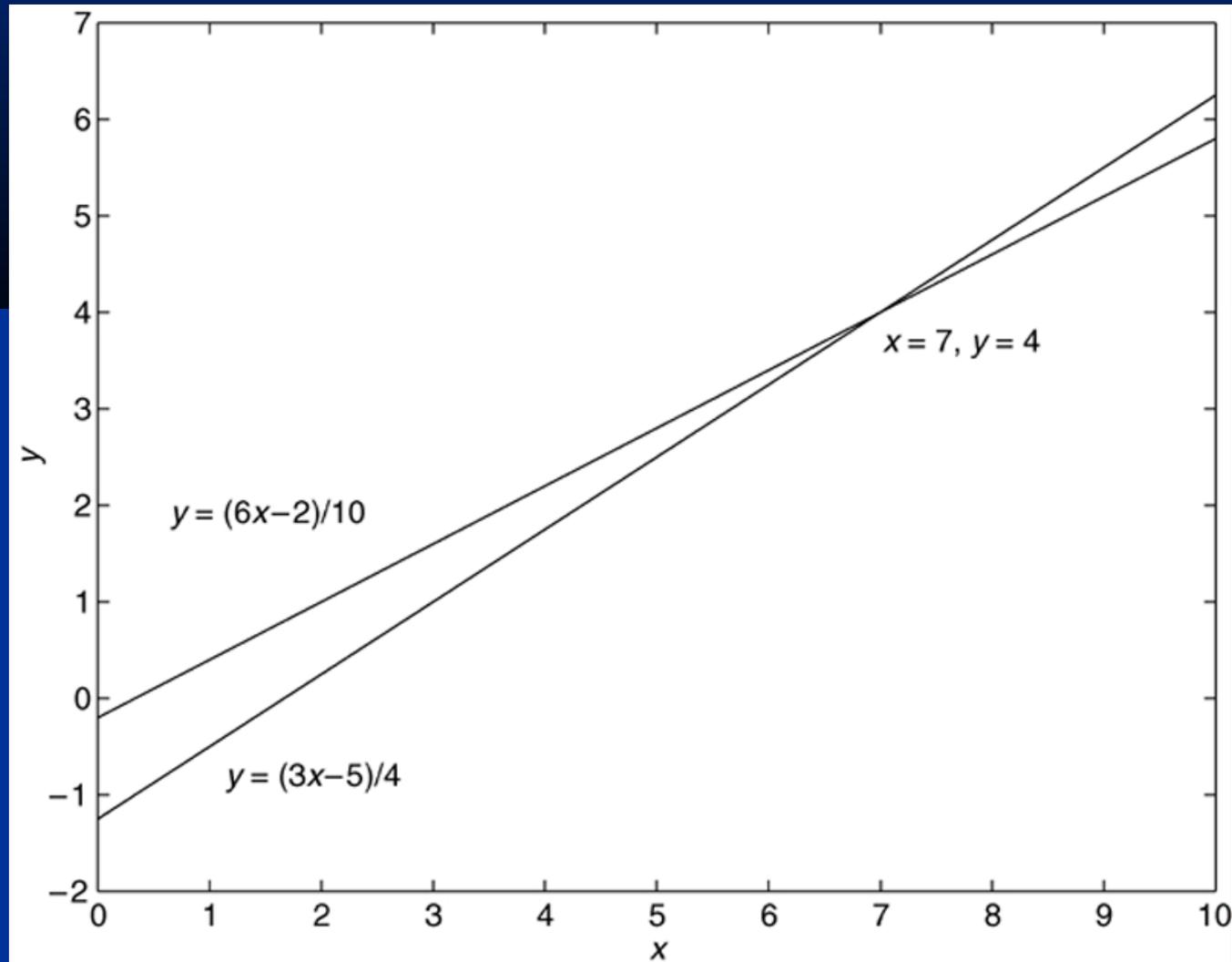
have graphs that intersect at the solution  $y = 4, x = 7$ .  
On the other hand, the set

$$3x - 4y = 5$$

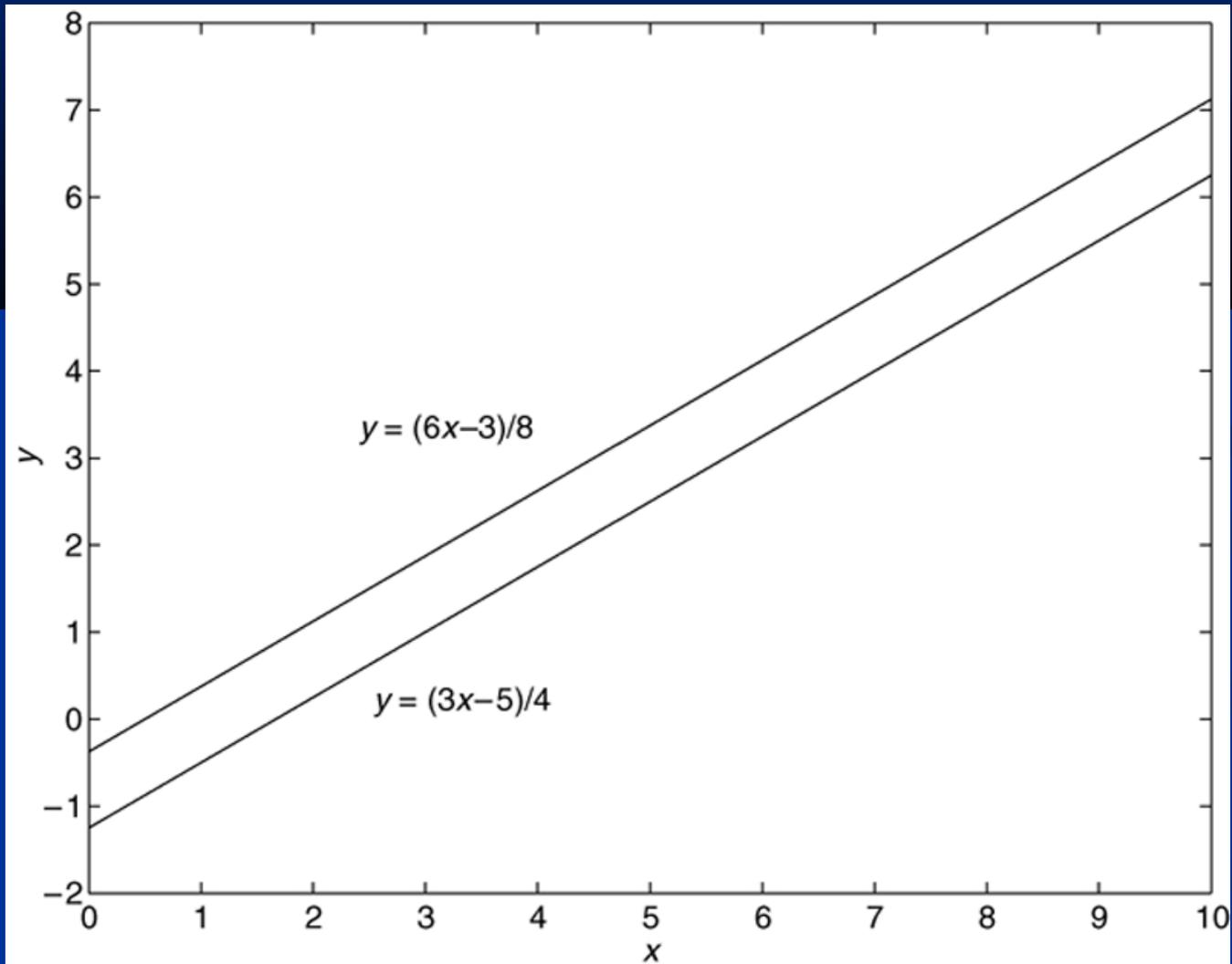
$$6x - 8y = 3$$

is singular but has no solution. The graphs of these two equations are distinct but *parallel* (see the next slide). Because they do not intersect, no solution exists.

The graphs of two equations that intersect at a solution.



Parallel graphs indicate that no solution exists.



Consider the following set of *homogeneous equations* (which means that their right sides are all zero)

$$6x + ay = 0$$

$$2x + 4y = 0$$

where  $a$  is a parameter. Multiply the second equation by 3 and subtract the result from the first equation to obtain

$$(a - 12)y = 0$$

The solution is  $y = 0$  *only if*  $a \neq 12$ ; if  $a = 12$ , there is an infinite number of solutions for  $x$  and  $y$ , where  $x = -2y$ .

Matrix notation enables us to represent multiple equations as a single matrix equation. For example, consider the following set:

$$2x_1 + 9x_2 = 5$$

$$3x_1 - 4x_2 = 7$$

This set can be expressed in vector-matrix form as

$$\begin{bmatrix} 2 & 9 \\ 3 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \end{bmatrix}$$

which can be represented in the following compact form

$$\mathbf{Ax} = \mathbf{b} \quad (\text{see pages 332-333})$$

For the equation set  $\mathbf{Ax} = \mathbf{b}$ ,

- if  $|\mathbf{A}| = 0$ , then there is no unique solution.
- Depending on the values in the vector  $\mathbf{b}$ , there may be no solution at all, or an infinite number of solutions.

The MATLAB command `inv(A)` computes the inverse of the matrix **A**. The following MATLAB session solves the following equations using MATLAB.

$$\begin{aligned}2x + 9y &= 5 \\3x - 4y &= 7\end{aligned}$$

```
>>A = [2,9;3,-4];b = [5;7]
>>x = inv(A)*b
x =
    2.3714
    0.0286
```

If you attempt to solve a singular problem using the `inv` command, MATLAB displays an error message.

## Existence and uniqueness of solutions (pages 334-335)

The set  $\mathbf{Ax} = \mathbf{b}$  with  $m$  equations and  $n$  unknowns has solutions if and only if

$$\text{rank}[\mathbf{A}] = \text{rank}[\mathbf{A} \ \mathbf{b}] \quad (1)$$

Let  $r = \text{rank}[\mathbf{A}]$ .

- If condition (1) is satisfied and if  $r = n$ , then the solution is unique.
- If condition (1) is satisfied but  $r < n$ , an infinite number of solutions exists and  $r$  unknown variables can be expressed as linear combinations of the other  $n - r$  unknown variables, whose values are arbitrary.

## Homogeneous case.

The homogeneous set  $\mathbf{Ax} = \mathbf{0}$  is a special case in which  $\mathbf{b} = \mathbf{0}$ .

For this case  $\text{rank}[\mathbf{A}] = \text{rank}[\mathbf{A} \ \mathbf{b}]$  always, and thus the set always has the trivial solution  $\mathbf{x} = \mathbf{0}$ .

A nonzero solution, in which at least one unknown is nonzero, exists if and only if  $\text{rank}[\mathbf{A}] < n$ .

If  $m < n$ , the homogeneous set always has a nonzero solution.

If the number of equations equals the number of unknowns and if  $|\mathbf{A}| \neq 0$ , then the equation set has a solution and it is unique.

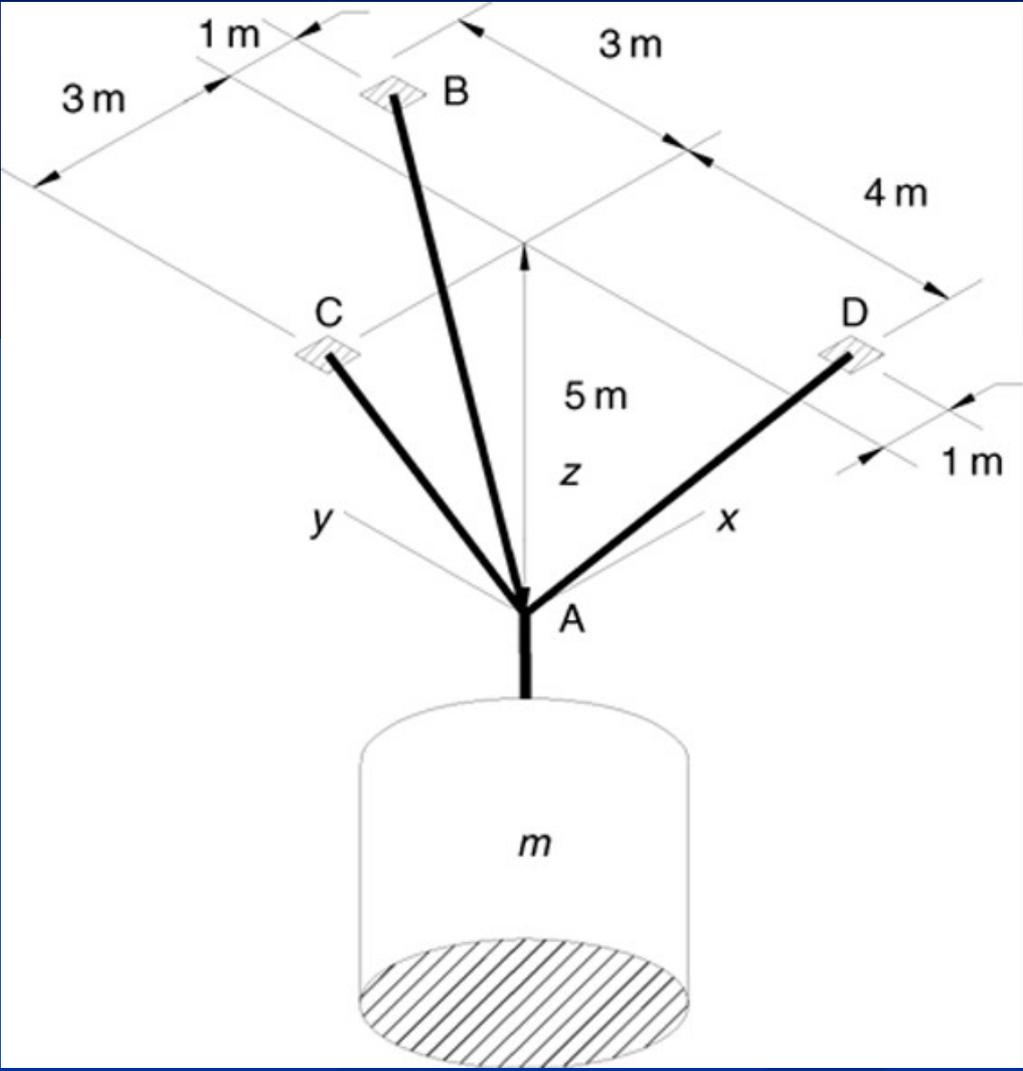
If  $|\mathbf{A}| = 0$  or if the number of equations does not equal the number of unknowns, then you must use the methods presented in Sections 8.3 or 8.4.

MATLAB provides the *left-division* method for solving the equation set  $\mathbf{Ax} = \mathbf{b}$ . The left-division method is based on Gauss elimination. (Section 8.2 on page 335) To use the left-division method to solve for  $\mathbf{x}$ , type  $\mathbf{x} = \mathbf{A}\backslash\mathbf{b}$ . For example,

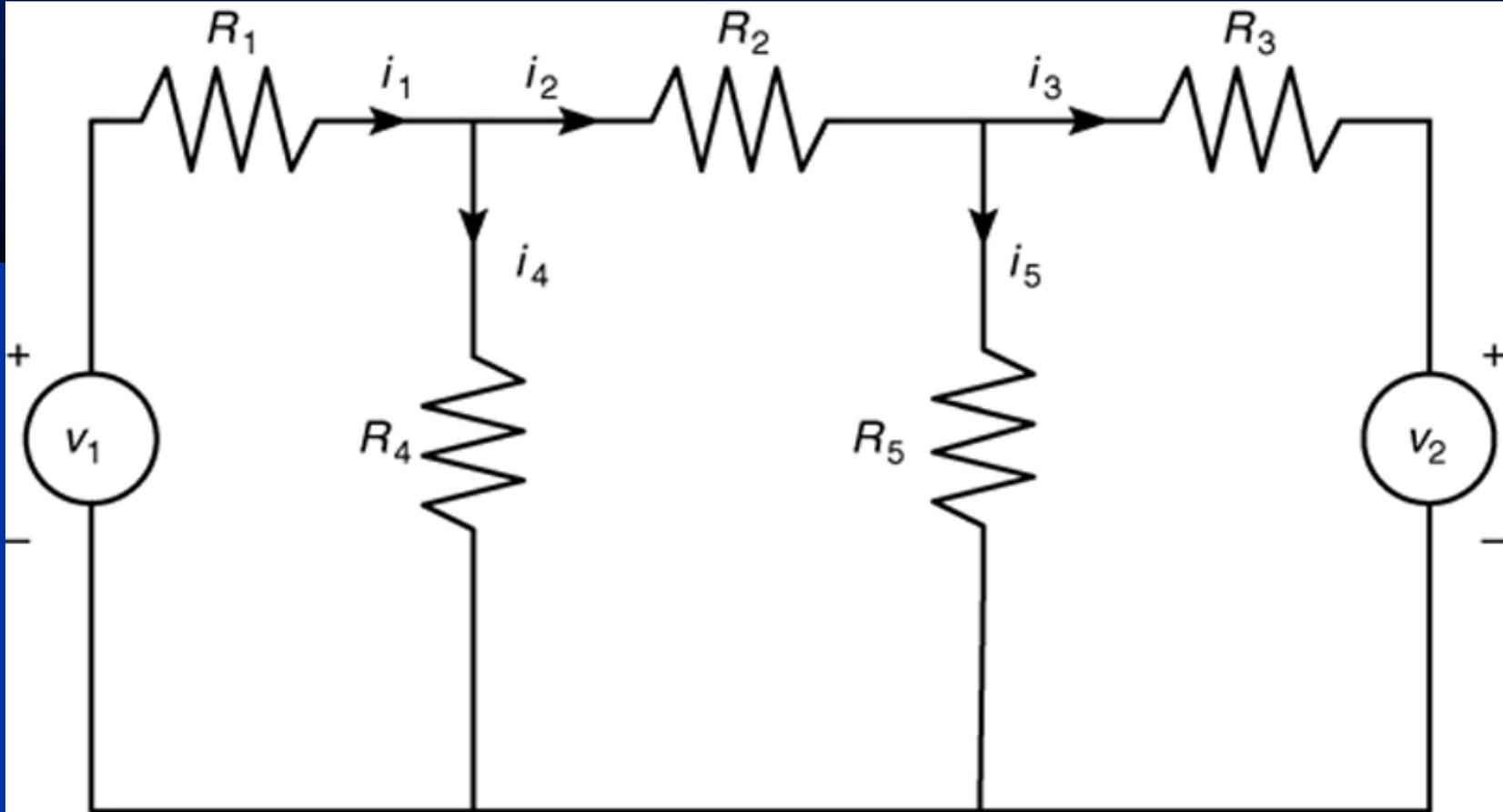
```
>> A = [6, -10; 3, -4]; b = [2; 5];  
>> x = A\b  
x =  
    7    4
```

This method also works in some cases where the number of unknowns does not equal the number of equations.

**Application of linear equations: Calculating cable tension for a suspended mass. Example 8.2-2. Figure 8.2-1, page 337.**



Application of linear equations: An electrical-resistance network. Example 8.2-3. Figure 8.2-2 on page 338



## Underdetermined Systems (Section 8.3 on page 341)

An *underdetermined system* does not contain enough information to solve for all of the unknown variables, usually because it has fewer equations than unknowns.

Thus an infinite number of solutions can exist, with one or more of the unknowns dependent on the remaining unknowns.

A simple example of an underdetermined systems is

$$x + 3y = 6$$

All we can do is solve for one of the unknowns in terms of the other; for example,  $x = 6 - 3y$ . An infinite number of solutions satisfy this equation.

When there are more equations than unknowns, the left-division method will give a solution with some of the unknowns set equal to zero. For example,

```
>>A = [1, 3]; b = 6;
```

```
>>solution = A\b
```

```
solution =
```

```
    0
```

```
    2
```

which corresponds to  $x = 0$  and  $y = 2$ .

An infinite number of solutions might exist even when the number of equations equals the number of unknowns.

This situation can occur when  $|\mathbf{A}| = 0$ .

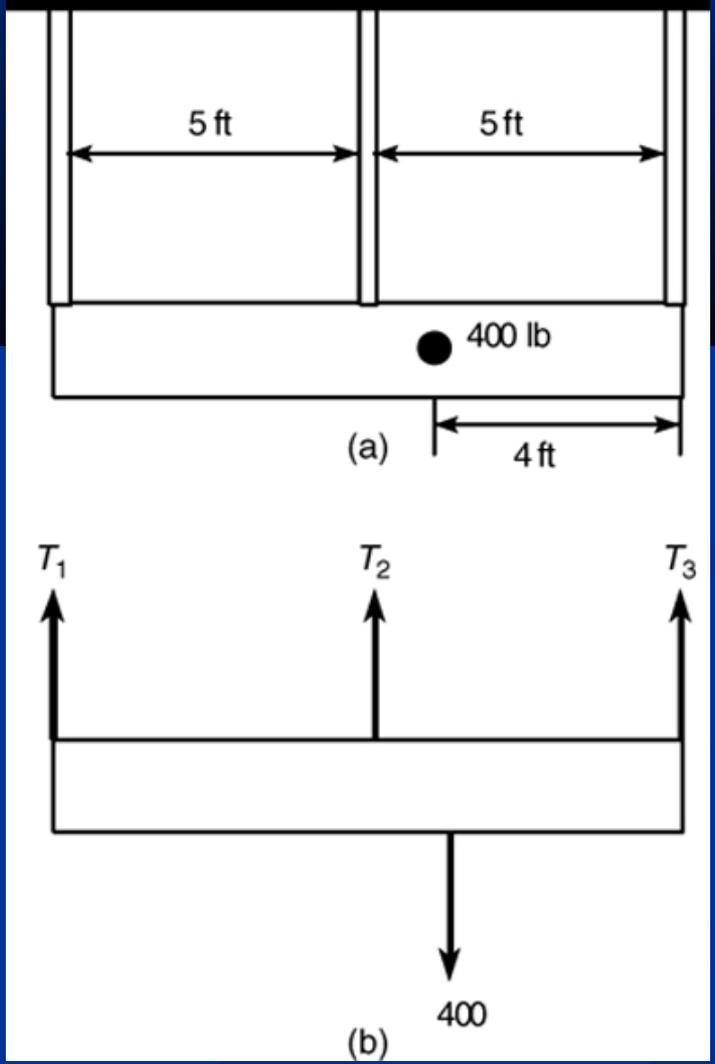
For such systems the matrix inverse method and Cramer's method will not work, and the left-division method generates an error message warning us that the matrix  $\mathbf{A}$  is singular.

In such cases the *pseudoinverse method*

$$\mathbf{x} = \text{pinv}(\mathbf{A}) * \mathbf{b}$$

gives one solution, the *minimum norm solution*.

**A Statically indeterminate problem. A light fixture and its free-body diagram. Example 8.3-2. Figure 8.3-1, page 344.**



In cases that have an infinite number of solutions, some of the unknowns can be expressed in terms of the remaining unknowns, whose values are arbitrary.

We can use the `rref` command to find these relations. See slide 8-24.

Recall that if  $|A| = 0$ , the equation set is singular.

If you try to solve a singular set using MATLAB, it prints a message warning that the matrix is singular and does not try to solve the problem.

An *ill-conditioned* set of equations is a set that is close to being singular.

The ill-conditioned status depends on the accuracy with which the solution calculations are made.

When the internal numerical accuracy used by MATLAB is insufficient to obtain a solution, MATLAB prints a message to warn you that the matrix is close to singular and that the results might be inaccurate.

The `pinv` command can obtain a solution of an underdetermined set.

To solve the equation set  $\mathbf{Ax} = \mathbf{b}$  using the `pinv` command, type

$$\mathbf{x} = \text{pinv}(\mathbf{A}) * \mathbf{b}$$

Underdetermined sets have an infinite number of solutions, and the `pinv` command produces a solution that gives the minimum value of the *Euclidean norm*, which is the magnitude of the solution vector  $\mathbf{x}$ .

## The `rref` function.

We can always express some of the unknowns in an underdetermined set as functions of the remaining unknowns. We can obtain such a form by multiplying the set's equations by suitable factors and adding the resulting equations to eliminate an unknown variable.

The MATLAB `rref` function provides a procedure to reduce an equation set to this form, which is called the *reduced row echelon form*.

The syntax is `rref([A b])`. The output is the augmented matrix  $[C \ d]$  that corresponds to the equation set  $Cx = d$ . This set is in reduced row echelon form.

## Overdetermined Systems (Section 8.4 on page 350)

An *overdetermined system* is a set of equations that has more independent equations than unknowns.

For such a system the matrix inverse method will not work because the **A** matrix is not square.

However, some overdetermined systems have exact solutions, and they can be obtained with the left division method  $x = A \backslash b$ .

For other overdetermined systems, no exact solution exists.

In some of these cases, the left-division method does not yield an answer, while in other cases the left-division method gives an answer that satisfies the equation set only in a “least squares” sense, as explained in Example 8.4–1 on pages 351-353.

When MATLAB gives an answer to an overdetermined set, it does not tell us whether the answer is the exact solution.

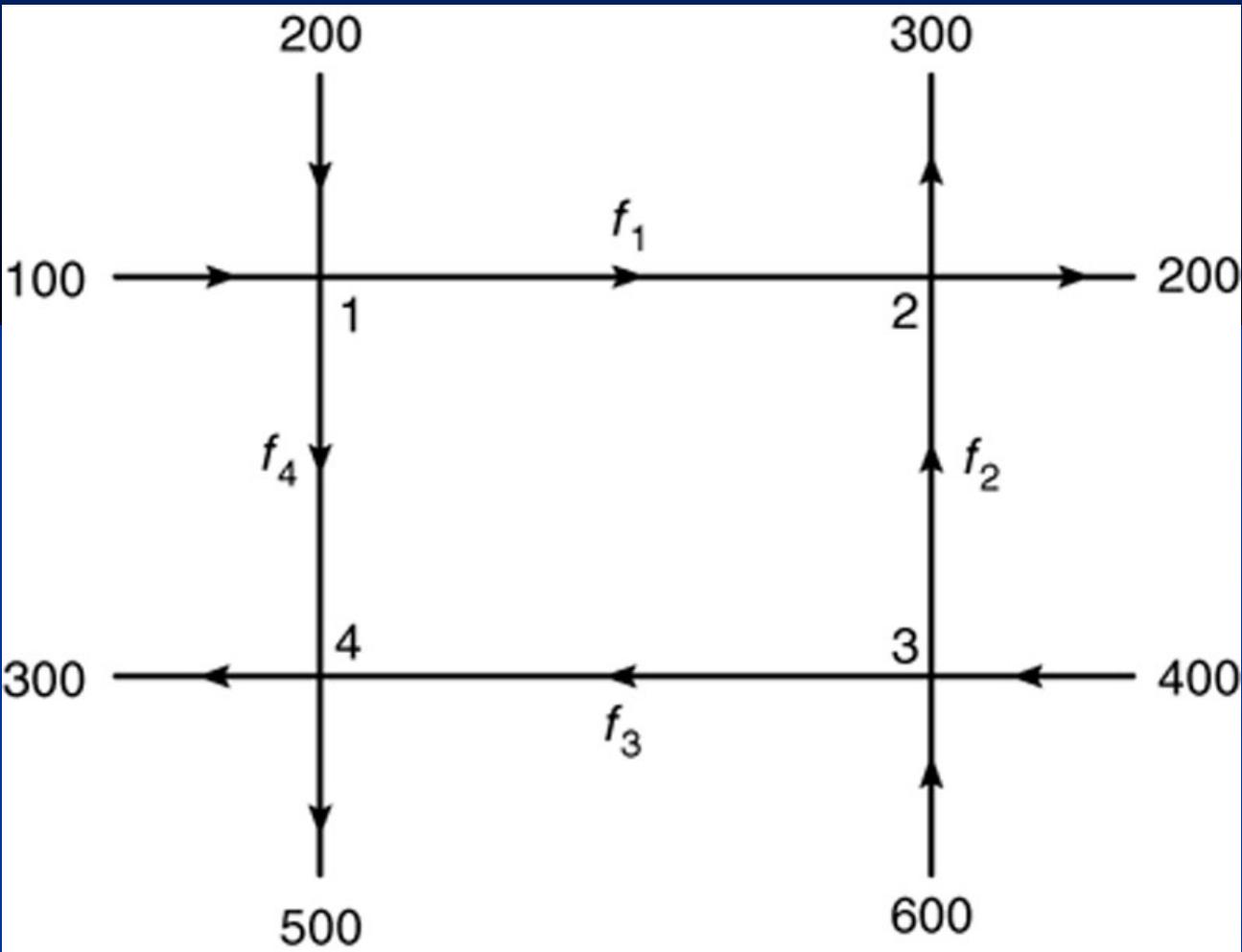
Some overdetermined systems have an exact solution.

The left-division method sometimes gives an answer for overdetermined systems, but it does not indicate whether the answer is the exact solution.

We need to check the ranks of  $\mathbf{A}$  and  $[\mathbf{A} \ \mathbf{b}]$  to know whether the answer is the exact solution.

To interpret MATLAB answers correctly for an overdetermined system, first check the ranks of  $A$  and  $[A \ b]$  to see whether an exact solution exists; if one does not exist, then you know that the left-division answer is a least squares solution.

Example of an underdetermined system. A network of one-way streets. Example 8.3-5. Figure 8.3-2, page 349.



## Solving Linear Equations: Summary

If the number of equations in the set *equals* the number of unknown variables, the matrix **A** is square and MATLAB provides two ways of solving the equation set **Ax = b**:

1. The matrix inverse method; solve for **x** by typing `x = inv(A)*b`.
2. The matrix left-division method; solve for **x** by typing `x = A\b`.

(continued ...)

## Solving Linear Equations: Summary (continued)

If  $A$  is square and if MATLAB does not generate an error message when you use one of these methods, then the set has a unique solution, which is given by the left-division method.

You can always check the solution for  $x$  by typing  $A * x$  to see if the result is the same as  $b$ .

(continued ...)

## Solving Linear Equations: Summary (continued)

If you receive an error message, the set is underdetermined, and either it does not have a solution or it has more than one solution.

In such a case, if you need more information, you must use the following procedures.

(continued ...)

## Solving Linear Equations: Summary (continued)

For underdetermined and overdetermined sets, MATLAB provides three ways of dealing with the equation set  $\mathbf{Ax} = \mathbf{b}$ . (Note that the matrix inverse method will never work with such sets.)

1. The matrix left-division method; solve for  $\mathbf{x}$  by typing  $\mathbf{x} = \mathbf{A} \backslash \mathbf{b}$ .
2. The pseudoinverse method; solve for  $\mathbf{x}$  by typing  $\mathbf{x} = \text{pinv}(\mathbf{A}) * \mathbf{b}$ .
3. The reduced row echelon form (RREF) method. This method uses the MATLAB function `rref` to obtain a solution.

## Solving Linear Equations: Summary (continued)

### Underdetermined Systems

In an *underdetermined* system not enough information is given to determine the values of all the unknown variables.

- An infinite number of solutions might exist in which one or more of the unknowns are dependent on the remaining unknowns.
- For such systems the matrix inverse method will not work because either  $\mathbf{A}$  is not square or because  $|\mathbf{A}| = 0$ .

## Solving Linear Equations: Summary (continued)

- The left-division method will give a solution with some of the unknowns arbitrarily set equal to zero, but this solution is not the general solution.
- An infinite number of solutions might exist even when the number of equations equals the number of unknowns. The left-division method fails to give a solution in such cases.
- In cases that have an infinite number of solutions, some of the unknowns can be expressed in terms of the remaining unknowns, whose values are arbitrary. The `rref` function can be used to find these relations.

(continued ...)

# Solving Linear Equations: Summary (continued)

## Overdetermined Systems

An *overdetermined* system is a set of equations that has more independent equations than unknowns.

- For such a system Cramer's method and the matrix inverse method will not work because the  $A$  matrix is not square.
- Some overdetermined systems have exact solutions, which can be obtained with the left-division method  $A \setminus b$ .

(continued ...)

## Solving Linear Equations: Summary (continued)

- For overdetermined systems that have no exact solution, the answer given by the left-division method satisfies the equation set only in a least squares sense.

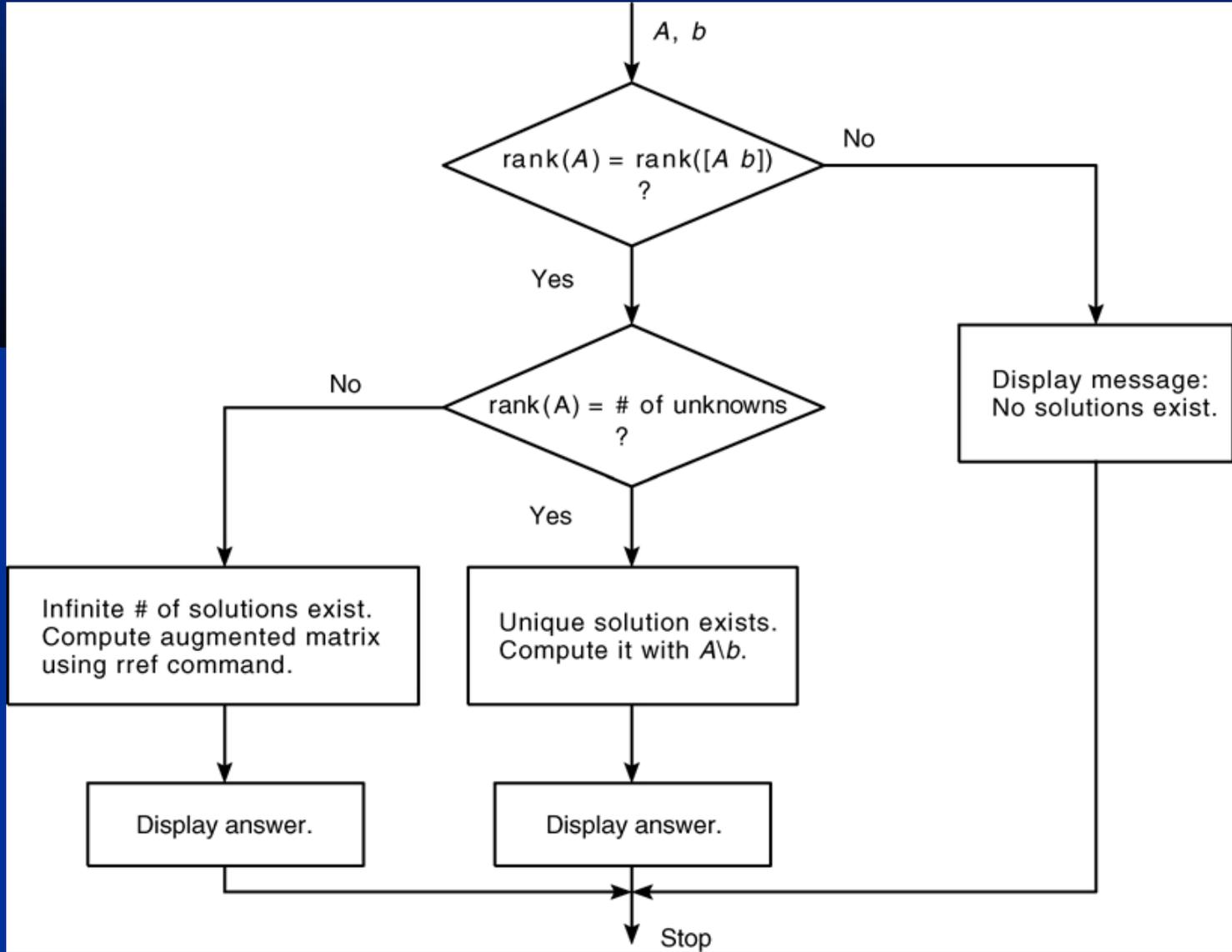
- When we use MATLAB to solve an overdetermined set, the program does not tell us whether the solution is exact. We must determine this information ourselves. The first step is to check the ranks of  $\mathbf{A}$  and  $[\mathbf{A} \ \mathbf{b}]$  to see whether a solution exists; if no solution exists, then we know that the left-division solution is a least squares answer.

More? See pages 398-402.

## Pseudocode for the linear equation solver. Table 8.5–1, page 354.

1. If the rank of  $\mathbf{A}$  equals the rank of  $[\mathbf{A} \ \mathbf{b}]$ , then determine whether the rank of  $\mathbf{A}$  equals the number of unknowns. If so, there is a unique solution, which can be computed using left division. Display the results and stop.
2. Otherwise, there is an infinite number of solutions, which can be found from the augmented matrix. Display the results and stop.
3. Otherwise (if the rank of  $\mathbf{A}$  does not equal the rank of  $[\mathbf{A} \ \mathbf{b}]$ ), then there are no solutions. Display this message and stop.

# Flowchart of the linear equation solver. Figure 8.5-1



## MATLAB program to solve linear equations. Table 8.5-2

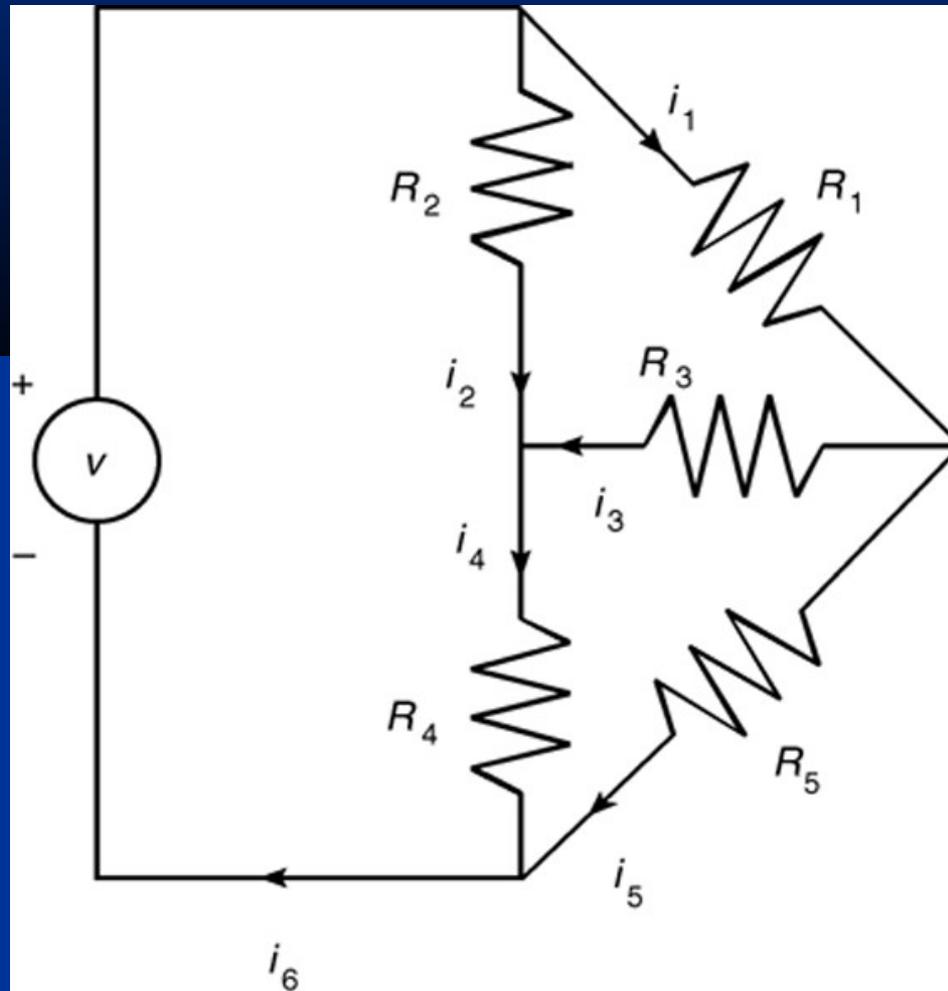
```
% Script file lineq.m
% Solves the set  $Ax = b$ , given A and b.
% Check the ranks of A and [A b].
if rank(A) == rank([A b])
    % The ranks are equal.
    Size_A = size(A);
    % Does the rank of A equal the number of
        unknowns?
    if rank(A) == size_A(2)
        % Yes. Rank of A equals the number of
            unknowns.
        disp('There is a unique solution,
                which is:')
        x = A\b % Solve using left division.
```

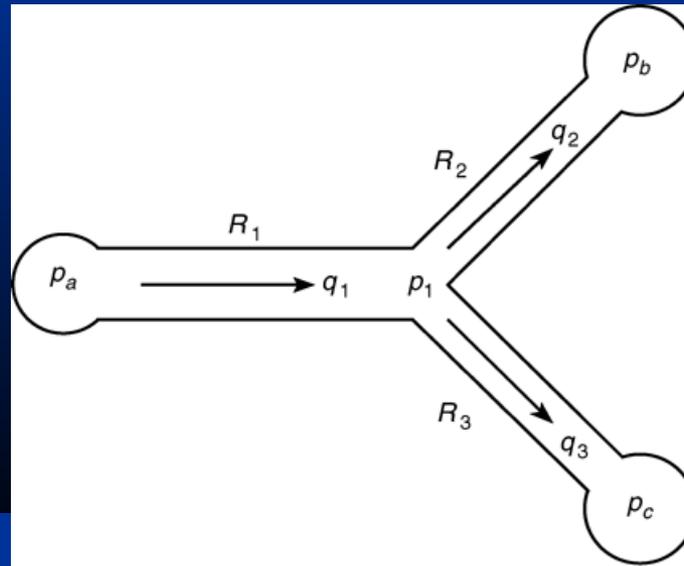
## Linear equation solver (continued)

```
else
    % Rank of A does not equal the number
    % of unknowns.
    disp('There is an infinite number of
    solutions.')
    disp('The augmented matrix of the
    reduced system is:')
    rref([A b]) % Compute the augmented
    matrix.
end
else
    % The ranks of A and [A b] are not equal.
    disp('There are no solutions.')
end
```

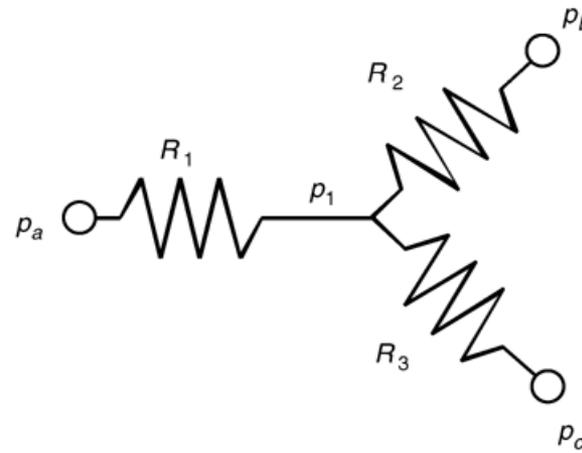
The following slides contain figures from the chapter's homework problems.

Figure P4, page 358





(a)



(b)

Figure P7, page 361

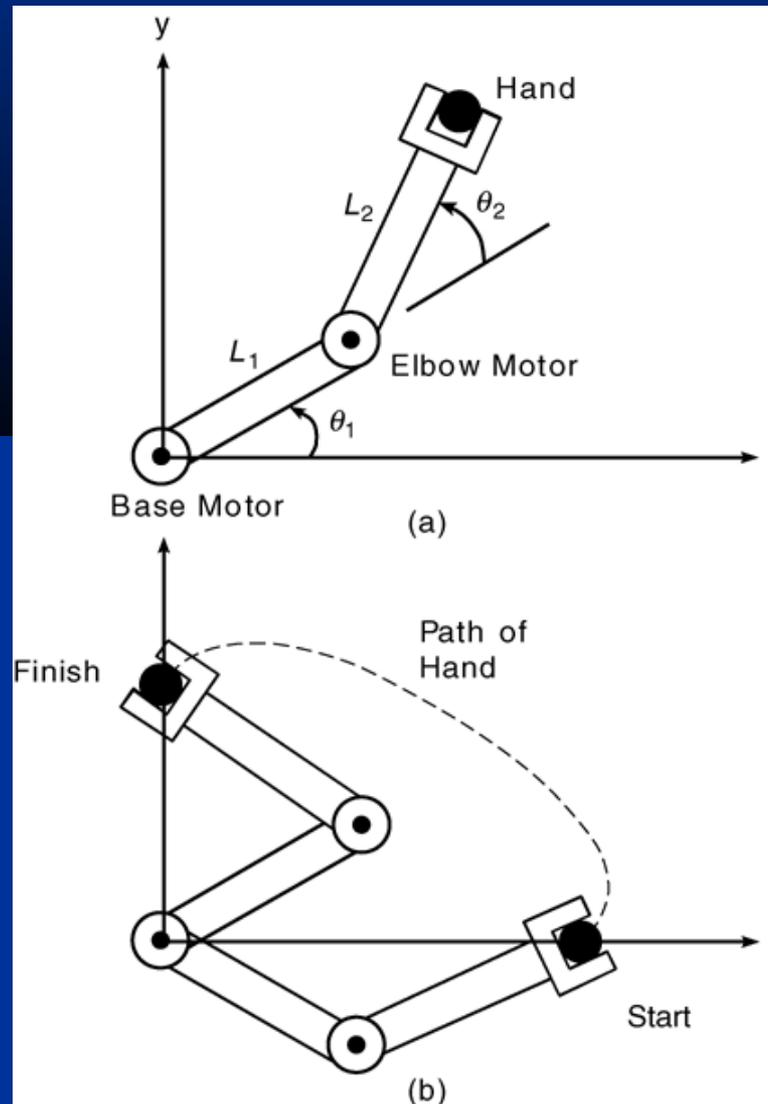
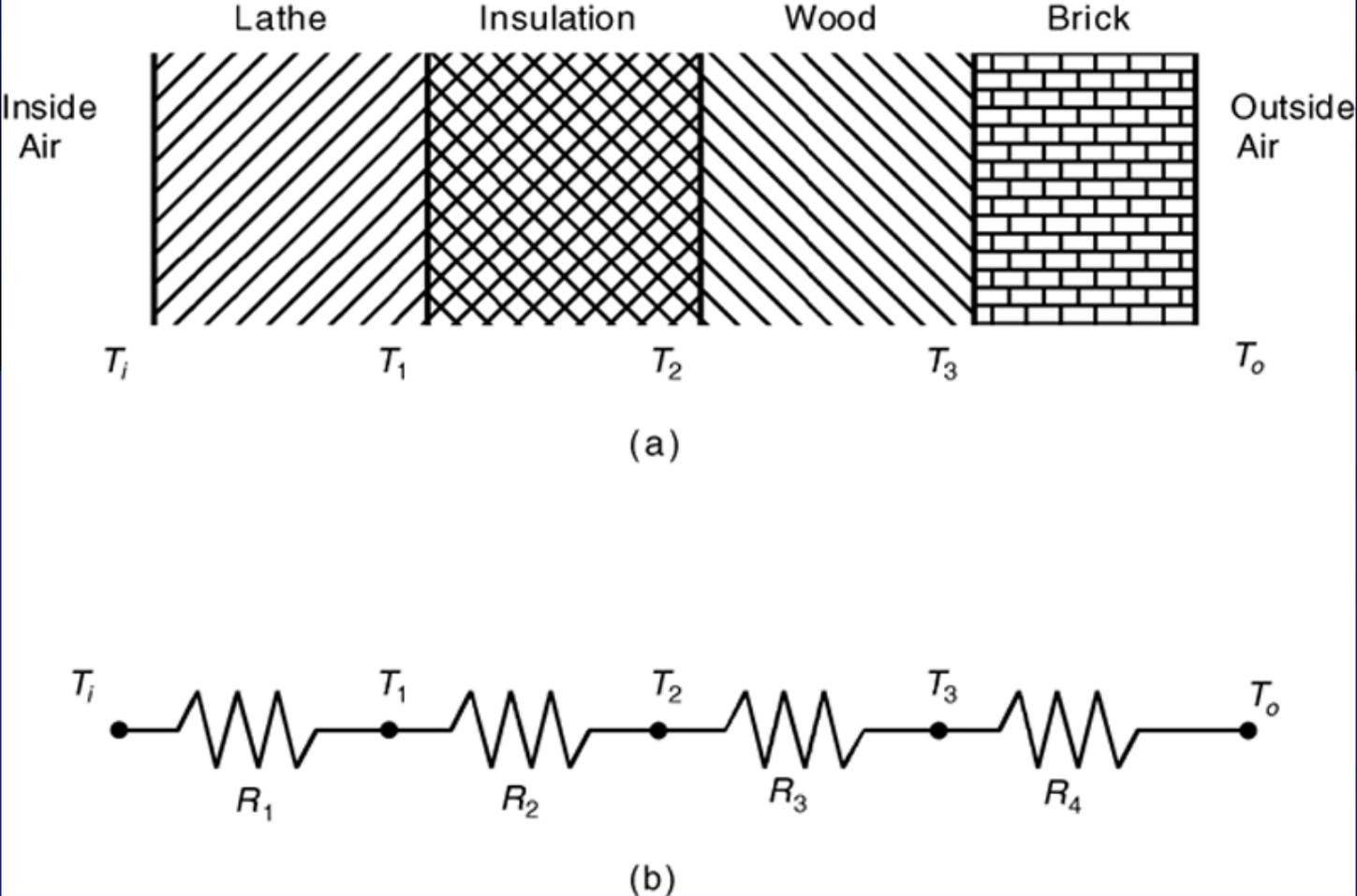


Figure P8, page 362



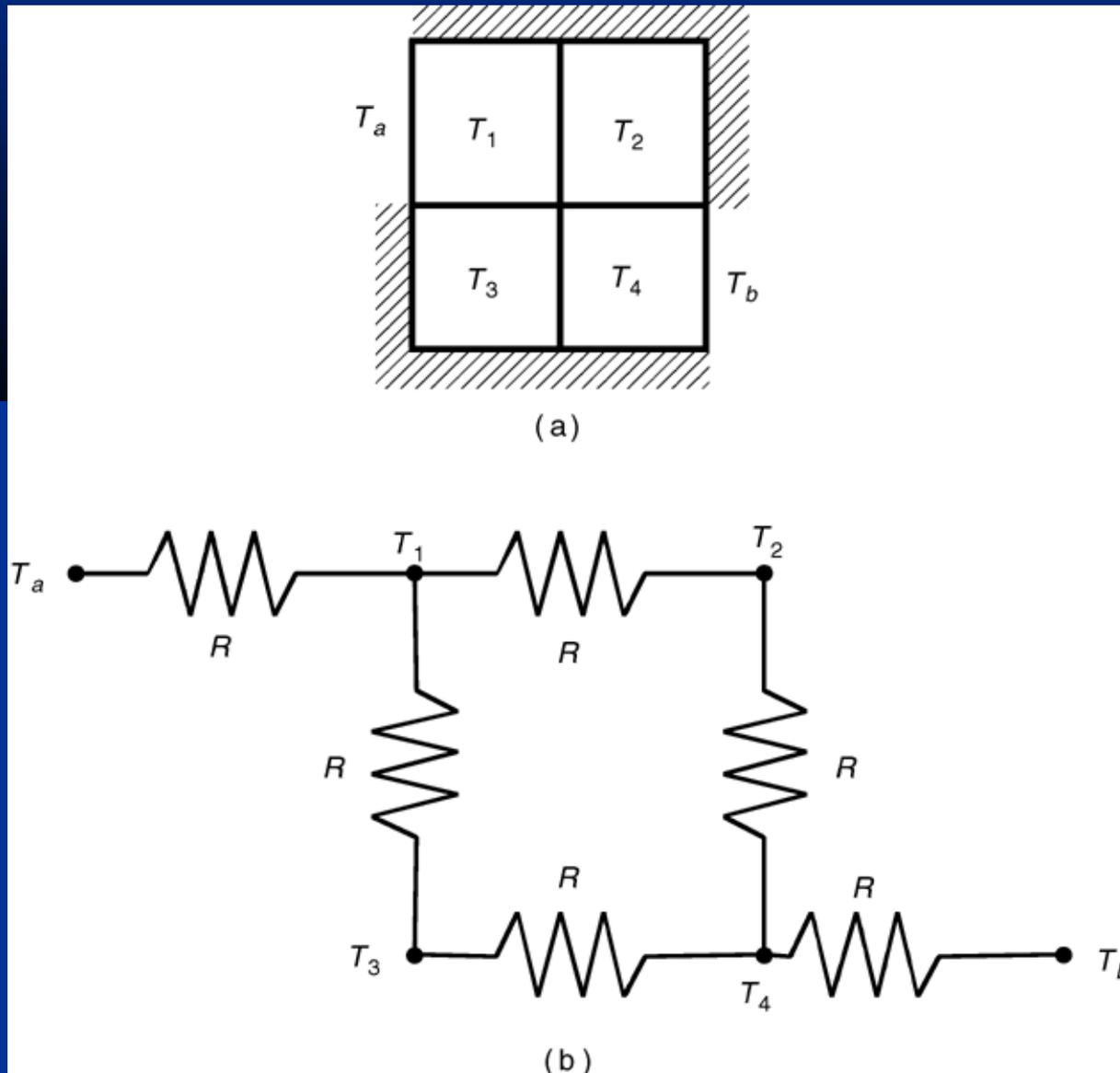


Figure P10, page 365

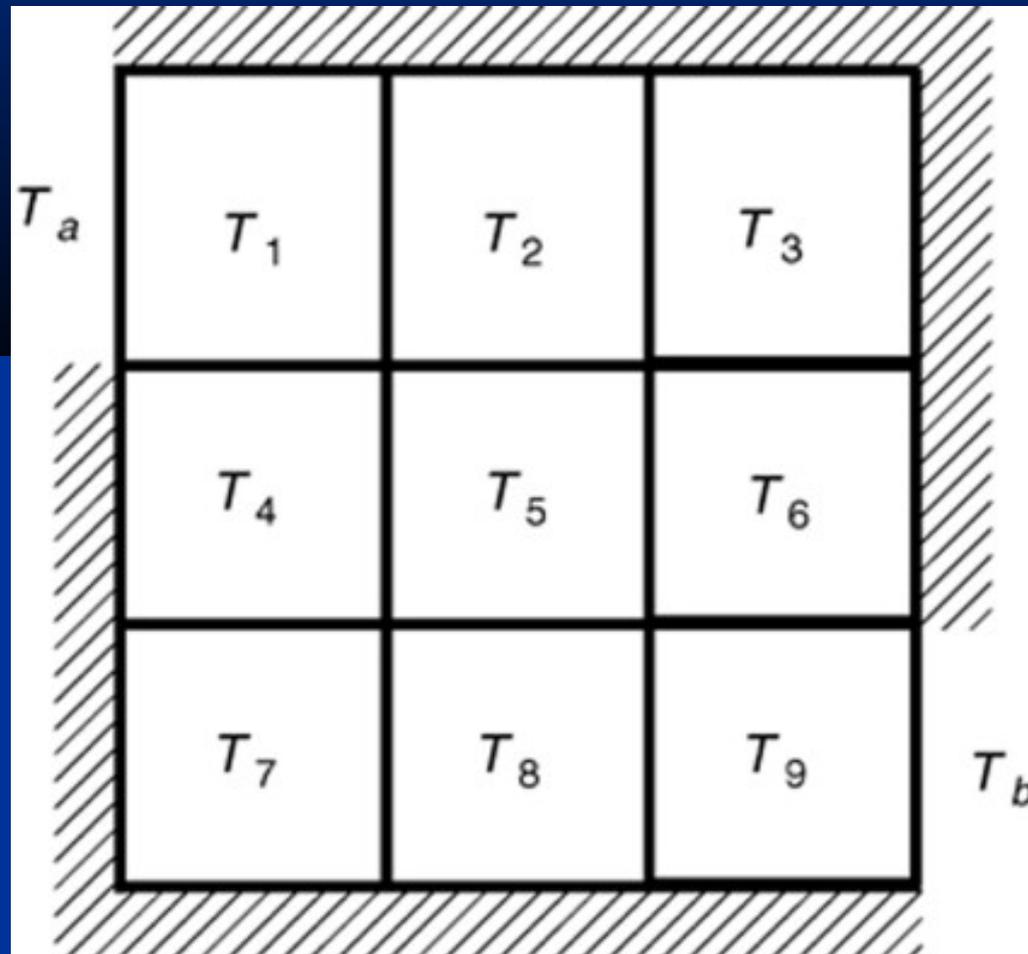


Figure P13, page 366

