Chapter 1

Fundamental Concepts

"Science knows no country because knowledge belongs to humanity and is the torch which illuminates the world. Science is the highest personification of the nation because that nation will remain the first which carries the furthest the works of thoughts and intelligence." Louis Pasteur

1.1 Introduction

Scientists and engineers use several techniques in solving continuum or field problems. Loosely speaking, these techniques can be classified as experimental, analytical, or numerical. Experiments are expensive, time consuming, sometimes hazardous, and usually do not allow much flexibility in parameter variation. However, every numerical method, as we shall see, involves an analytic simplification to the point where it is easy to apply the numerical method. Notwithstanding this fact, the following methods are among the most commonly used in electromagnetics (EM).

A. Analytical methods (exact solutions)

(1) separation of variables
(2) series expansion
(3) conformal mapping
(4) integral solutions, e.g., Laplace and Fourier transforms
(5) perturbation methods

B. Numerical methods (approximate solutions)

(1) finite difference method
(2) method of weighted residuals
(3) moment method
(4) finite element method

(5) transmission-line modeling

(6) Monte Carlo method

(7) method of lines

Application of these methods is not limited to EM-related problems; they find applications in other continuum problems such as in fluid, heat transfer, and acoustics [1].

As we shall see, some of the numerical methods are related and they all generally give approximate solutions of sufficient accuracy for engineering purposes. Since our objective is to study these methods in detail in the subsequent chapters, it may be premature to say more than this at this point.

The need for numerical solution of electromagnetic problems is best expressed in the words of Paris and Hurd: "Most problems that can be solved formally (analytically) have been solved."[1] Until the 1940s, most EM problems were solved using the classical methods of separation of variables and integral equation solutions. Besides the fact that a high degree of ingenuity, experience, and effort were required to apply those methods, only a narrow range of practical problems could be investigated due to the complex geometries defining the problems.

Numerical solution of EM problems started in the mid-1960s with the availability of modern high-speed digital computers. Since then, considerable effort has been expended on solving practical, complex EM-related problems for which closed form analytical solutions are either intractable or do not exist. The numerical approach has the advantage of allowing the actual work to be carried out by operators without a knowledge of higher mathematics or physics, with a resulting economy of labor on the part of the highly trained personnel.

Before we set out to study the various techniques used in analyzing EM problems, it is expedient to remind ourselves of the physical laws governing EM phenomena in general. This will be done in Section 1.2. In Section 1.3, we shall be acquainted with different ways EM problems are categorized. The principle of superposition and uniqueness theorem will be covered in Section 1.4.

1.2 Review of Electromagnetic Theory

The whole subject of EM unfolds as a logical deduction from eight postulated equations, namely, Maxwell's four field equations and four medium-dependent equations [2]–[4]. Before we briefly review these equations, it may be helpful to state two important theorems commonly used in EM. These are the divergence (or Gauss's)

[1] *Basic Electromagnetic Theory,* D.T. Paris and F.K. Hurd, McGraw-Hill, New York, 1969, p. 166.

theorem,

$$\oint_S \mathbf{F} \cdot d\mathbf{S} = \int_v \nabla \cdot \mathbf{F} \, dv \tag{1.1}$$

and Stokes's theorem

$$\oint_L \mathbf{F} \cdot d\mathbf{l} = \int_S \nabla \times \mathbf{F} \cdot d\mathbf{S} \tag{1.2}$$

Perhaps the best way to review EM theory is by using the fundamental concept of electric charge. EM theory can be regarded as the study of fields produced by electric charges at rest and in motion. Electrostatic fields are usually produced by static electric charges, whereas magnetostatic fields are due to motion of electric charges with uniform velocity (direct current). Dynamic or time-varying fields are usually due to accelerated charges or time-varying currents.

1.2.1 Electrostatic Fields

The two fundamental laws governing these electrostatic fields are Gauss's law,

$$\oint \mathbf{D} \cdot d\mathbf{S} = \int \rho_v \, dv \tag{1.3}$$

which is a direct consequence of Coulomb's force law, and the law describing electrostatic fields as conservative,

$$\oint \mathbf{E} \cdot d\mathbf{l} = 0 \tag{1.4}$$

In Eqs. (1.3) and (1.4), $\mathbf{D}$ is the electric flux density (in coulombs/meter^2), ρ_v is the volume charge density (in coulombs/meter^3), and $\mathbf{E}$ is the electric field intensity (in volts/meter). The integral form of the laws in Eqs. (1.3) and (1.4) can be expressed in the differential form by applying Eq. (1.1) to Eq. (1.3) and Eq. (1.2) to Eq. (1.4). We obtain

$$\nabla \cdot \mathbf{D} = \rho_v \tag{1.5}$$

and

$$\nabla \times \mathbf{E} = 0 \tag{1.6}$$

The vector fields $\mathbf{D}$ and $\mathbf{E}$ are related as

$$\mathbf{D} = \epsilon \mathbf{E} \tag{1.7}$$

where ϵ is the dielectric permittivity (in farads/meter) of the medium. In terms of the electric potential V (in volts), $\mathbf{E}$ is expressed as

$$\mathbf{E} = -\nabla V \tag{1.8}$$

or

$$V = -\int \mathbf{E} \cdot d\mathbf{l} \tag{1.9}$$

Combining Eqs. (1.5), (1.7), and (1.8) gives Poisson's equation:

$$\nabla \cdot \epsilon \nabla V = -\rho_v \tag{1.10a}$$

or, if ϵ is constant,

$$\boxed{\nabla^2 V = -\frac{\rho_v}{\epsilon}} \tag{1.10b}$$

When $\rho_v = 0$, Eq. (1.10) becomes Laplace's equation:

$$\nabla \cdot \epsilon \nabla V = 0 \tag{1.11a}$$

or for constant ϵ

$$\boxed{\nabla^2 V = 0} \tag{1.11b}$$

1.2.2 Magnetostatic Fields

The basic laws of magnetostatic fields are Ampere's law

$$\oint_L \mathbf{H} \cdot d\mathbf{l} = \int_S \mathbf{J} \cdot d\mathbf{S} \tag{1.12}$$

which is related to Biot-Savart law, and the law of conservation of magnetic flux (also called Gauss's law for magnetostatics)

$$\oint \mathbf{B} \cdot d\mathbf{S} = 0 \tag{1.13}$$

where $\mathbf{H}$ is the magnetic field intensity (in amperes/meter), $\mathbf{J}_e$ is the electric current density (in amperes/meter^2), and $\mathbf{B}$ is the magnetic flux density (in tesla or webers/meter^2). Applying Eq. (1.2) to Eq. (1.12) and Eq. (1.1) to Eq. (1.13) yields their differential form as

$$\nabla \times \mathbf{H} = \mathbf{J}_e \tag{1.14}$$

and

$$\nabla \cdot \mathbf{B} = 0 \tag{1.15}$$

The vector fields $\mathbf{B}$ and $\mathbf{H}$ are related through the permeability μ (in henries/meter) of the medium as

$$\mathbf{B} = \mu \mathbf{H} \tag{1.16}$$

Also, $\mathbf{J}$ is related to $\mathbf{E}$ through the conductivity σ (in mhos/meter) of the medium as

$$\mathbf{J} = \sigma \mathbf{E} \tag{1.17}$$

This is usually referred to as point form of Ohm's law. In terms of the magnetic vector potential $\mathbf{A}$ (in Wb/meter)

$$\mathbf{B} = \nabla \times \mathbf{A} \tag{1.18}$$

Applying the vector identity

$$\nabla \times (\nabla \times \mathbf{F}) = \nabla(\nabla \cdot \mathbf{F}) - \nabla^2 \mathbf{F} \tag{1.19}$$

to Eqs. (1.14) and (1.18) and assuming Coulomb gauge condition ($\nabla \cdot \mathbf{A} = 0$) leads to Poisson's equation for magnetostatic fields:

$$\boxed{\nabla^2 \mathbf{A} = -\mu \mathbf{J}} \tag{1.20}$$

When $\mathbf{J} = 0$, Eq. (1.20) becomes Laplace's equation

$$\boxed{\nabla^2 \mathbf{A} = 0} \tag{1.21}$$

1.2.3 Time-varying Fields

In this case, electric and magnetic fields exist simultaneously. Equations (1.5) and (1.15) remain the same whereas Eqs. (1.6) and (1.14) require some modification for dynamic fields. Modification of Eq. (1.6) is necessary to incorporate Faraday's law of induction, and that of Eq. (1.14) is warranted to allow for displacement current. The time-varying EM fields are governed by physical laws expressed mathematically as

$$\nabla \cdot \mathbf{D} = \rho_v \tag{1.22a}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{1.22b}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} - \mathbf{J}_m \tag{1.22c}$$

$$\nabla \times \mathbf{H} = \mathbf{J}_e + \frac{\partial \mathbf{D}}{\partial t} \tag{1.22d}$$

where $\mathbf{J}_m = \sigma^* \mathbf{H}$ is the magnetic conductive current density (in volts/square meter) and σ^* is the magnetic resistivity (in ohms/meter).

These equations are referred to as Maxwell's equations in the generalized form. They are first-order linear coupled differential equations relating the vector field quantities to each other. The equivalent integral form of Eq. (1.22) is

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = \int_v \rho_v dv \tag{1.23a}$$

$$\oint_S \mathbf{B} \cdot d\mathbf{S} = 0 \tag{1.23b}$$

$$\oint_L \mathbf{E} \cdot d\mathbf{l} = -\int_S \left(\frac{\partial \mathbf{B}}{\partial t} + \mathbf{J}_m \right) \cdot d\mathbf{S} \tag{1.23c}$$

$$\oint_L \mathbf{H} \cdot d\mathbf{l} = \int_S \left(\mathbf{J}_e + \frac{\partial \mathbf{D}}{\partial t} \right) \cdot d\mathbf{S} \tag{1.23d}$$

In addition to these four Maxwell's equations, there are four medium-dependent equations:

$$\boxed{\begin{aligned} \mathbf{D} &= \epsilon \mathbf{E} && (1.24a) \\ \mathbf{B} &= \mu \mathbf{H} && (1.24b) \\ \mathbf{J}_e &= \sigma \mathbf{E} && (1.24c) \\ \mathbf{J}_m &= \sigma^* \mathbf{M} && (1.24d) \end{aligned}}$$

These are called *constitutive relations* for the medium in which the fields exist. Equations (1.22) and (1.24) form the eight postulated equations on which EM theory unfolds itself. We must note that in the region where Maxwellian fields exist, the fields are assumed to be:

(1) single valued,

(2) bounded, and

(3) continuous functions of space and time with continuous derivatives.

It is worthwhile to mention two other fundamental equations that go hand-in-hand with Maxwell's equations. One is the Lorentz force equation

$$\mathbf{F} = Q(\mathbf{E} + \mathbf{u} \times \mathbf{B}) \tag{1.25}$$

where $\mathbf{F}$ is the force experienced by a particle with charge Q moving at velocity $\mathbf{u}$ in an EM field; the Lorentz force equation constitutes a link between EM and mechanics. The other is the continuity equation

$$\boxed{\nabla \cdot \mathbf{J} = -\frac{\partial \rho_v}{\partial t}} \tag{1.26}$$

which expresses the conservation (or indestructibility) of electric charge. The continuity equation is implicit in Maxwell's equations (see Example 1.2). Equation (1.26) is not peculiar to EM. In fluid mechanics, where $\mathbf{J}$ corresponds with velocity and ρ_v with mass, Eq. (1.26) expresses the law of conservation of mass.

1.2.4 Boundary Conditions

The material medium in which an EM field exists is usually characterized by its constitutive parameters σ, ϵ, and μ. The medium is said to be *linear* if σ, ϵ, and μ are independent of **E** and **H** or nonlinear otherwise. It is *homogeneous* if σ, ϵ, and μ are not functions of space variables or inhomogeneous otherwise. It is *isotropic* if σ, ϵ, and μ are independent of direction (scalars) or anisotropic otherwise.

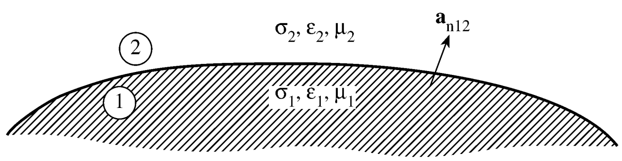

Figure 1.1
Interface between two media.

The boundary conditions at the interface separating two different media 1 and 2, with parameters $(\sigma_1, \epsilon_1, \mu_1)$ and $(\sigma_2, \epsilon_2, \mu_2)$ as shown in Fig. 1.1, are easily derived from the integral form of Maxwell's equations. They are

$$E_{1t} = E_{2t} \quad \text{or} \quad (\mathbf{E}_1 - \mathbf{E}_2) \times \mathbf{a}_{n12} = 0 \tag{1.27a}$$

$$H_{1t} - H_{21} = K \quad \text{or} \quad (\mathbf{H}_1 - \mathbf{H}_2) \times \mathbf{a}_{n12} = \mathbf{K} \tag{1.27b}$$

$$D_{1n} - D_{2n} = \rho_S \quad \text{or} \quad (\mathbf{D}_1 - \mathbf{D}_2) \cdot \mathbf{a}_{n12} = \rho_S \tag{1.27c}$$

$$B_{1n} - B_{2n} = 0 \quad \text{or} \quad (\mathbf{B}_2 - \mathbf{B}_1) \cdot \mathbf{a}_{n12} = 0 \tag{1.27d}$$

where $\mathbf{a}_{n12}$ is a unit normal vector directed from medium 1 to medium 2, subscripts 1 and 2 denote fields in regions 1 and 2, and subscripts t and n, respectively, denote tangent and normal components of the fields. Equations (1.27a) and (1.27d) state that the tangential components of **E** and the normal components of **B** are continuous across the boundary. Equation (1.27b) states that the tangential component of **H** is discontinuous by the surface current density **K** on the boundary. Equation (1.27c) states that the discontinuity in the normal component of **D** is the same as the surface charge density ρ_s on the boundary.

In practice, only two of Maxwell's equations are used (Eqs. (1.22c) and (1.22d)) when a medium is source-free ($\mathbf{J} = 0$, $\rho_v = 0$), since the other two are implied (see Problem 1.3). Also, in practice, it is sufficient to make the tangential components of the fields satisfy the necessary boundary conditions since the normal components implicitly satisfy their corresponding boundary conditions.

1.2.5 Wave Equations

As mentioned earlier, Maxwell's equations are coupled first-order differential equations which are difficult to apply when solving boundary-value problems. The difficulty is overcome by decoupling the first-order equations, thereby obtaining the wave equation, a second-order differential equation which is useful for solving problems.

To obtain the wave equation for a linear, isotropic, homogeneous, source-free medium ($\rho_v = 0$, $\mathbf{J} = 0$) from Eq. (1.22), we take the curl of both sides of Eq. (1.22c). This gives

$$\nabla \times \nabla \times \mathbf{E} = -\mu \frac{\partial}{\partial t}(\nabla \times \mathbf{H}) \tag{1.28}$$

From (1.22d),

$$\nabla \times \mathbf{H} = \epsilon \frac{\partial \mathbf{E}}{\partial t}$$

since $\mathbf{J} = 0$, so that Eq. (1.28) becomes

$$\nabla \times \nabla \times \mathbf{E} = -\mu\epsilon \frac{\partial^2 \mathbf{E}}{\partial t^2} \tag{1.29}$$

Applying the vector identity

$$\nabla \times \nabla \times \mathbf{F} = \nabla(\nabla \cdot \mathbf{F}) - \nabla^2 \mathbf{F} \tag{1.30}$$

in Eq. (1.29),

$$\nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E} = -\mu\epsilon \frac{\partial^2 \mathbf{E}}{\partial t^2}$$

Since $\rho_v = 0$, $\nabla \cdot \mathbf{E} = 0$ from Eq. (1.22a), and hence we obtain

$$\boxed{\nabla^2 \mathbf{E} - \mu\epsilon \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0} \tag{1.31}$$

which is the time-dependent vector Helmholtz equation or simply wave equation. If we had started the derivation with Eq. (1.22d), we would obtain the wave equation for $\mathbf{H}$ as

$$\boxed{\nabla^2 \mathbf{H} - \mu\epsilon \frac{\partial^2 \mathbf{H}}{\partial t^2} = 0} \tag{1.32}$$

Equations (1.31) and (1.32) are the equations of motion of EM waves in the medium under consideration. The velocity (in *m/s*) of wave propagation is

$$u = \frac{1}{\sqrt{\mu\epsilon}} \tag{1.33}$$

where $u = c \approx 3 \times 10^8$ *m/s* in free space. It should be noted that each of the vector equations in (1.31) and (1.32) has three scalar components, so that altogether we have six scalar equations for E_x, E_y, E_z, H_x, H_y, and H_z. Thus each component of the wave equations has the form

$$\nabla^2 \Psi - \frac{1}{u^2} \frac{\partial^2 \Psi}{\partial t^2} = 0 \tag{1.34}$$

which is the scalar wave equation.

1.2.6 Time-varying Potentials

Although we are often interested in electric and magnetic field intensities (**E** and **H**), which are physically measurable quantities, it is often convenient to use auxiliary functions in analyzing an EM field. These auxiliary functions are the scalar electric potential V and vector magnetic potential **A**. Although these potential functions are arbitrary, they are required to satisfy Maxwell's equations. Their derivation is based on two fundamental vector identities (see Prob. 1.1),

$$\nabla \times \nabla \Phi = 0 \tag{1.35}$$

and

$$\nabla \cdot \nabla \times \mathbf{F} = 0 \tag{1.36}$$

which an arbitrary scalar field Φ and vector field **F** must satisfy. Maxwell's equation (1.22b) along with Eq. (1.36) is satisfied if we define **A** such that

$$\boxed{\mathbf{B} = \nabla \times \mathbf{A}} \tag{1.37}$$

Substituting this into Eq. (1.22c) gives

$$-\nabla \times \left(\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} \right) = 0$$

Since this equation has to be compatible with Eq. (1.35), we can choose the scalar field V such that

$$\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} = -\nabla V$$

or

$$\boxed{\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t}} \tag{1.38}$$

Thus, if we knew the potential functions V and **A**, the fields **E** and **B** could be obtained from Eqs. (1.37) and (1.38). However, we still need to find the solution for the potential functions. Substituting Eqs. (1.37) and (1.38) into Eq. (1.22d) and assuming a linear, homogeneous medium,

$$\nabla \times \nabla \times \mathbf{A} = \mu \mathbf{J} + \epsilon \mu \frac{\partial}{\partial t} \left(-\nabla V - \frac{\partial \mathbf{A}}{\partial t} \right)$$

Applying the vector identity in Eq. (1.30) leads to

$$\nabla^2 \mathbf{A} - \nabla(\nabla \cdot \mathbf{A}) = -\mu \mathbf{J} + \mu\epsilon \nabla \frac{\partial^2 \mathbf{A}}{\partial t^2} + \mu\epsilon \nabla \frac{\partial V}{\partial t} \tag{1.39}$$

Substituting Eq. (1.38) into Eq. (1.22a) gives

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon} = -\nabla^2 V - \frac{\partial (\nabla \cdot \mathbf{A})}{\partial t}$$

or

$$\nabla^2 V + \frac{\partial}{\partial t} \nabla \cdot \mathbf{A} = -\frac{\rho_v}{\epsilon} \tag{1.40}$$

According to the Helmholtz theorem of vector analysis, a vector is uniquely defined if and only if both its curl and divergence are specified. We have only specified the curl of **A** in Eq. (1.37); we may choose the divergence of **A** so that the differential equations (1.39) and (1.40) have the simplest forms possible. We achieve this in the so-called *Lorentz condition:*

$$\nabla \cdot \mathbf{A} = -\mu\epsilon \frac{\partial V}{\partial t} \tag{1.41}$$

Incorporating this condition into Eqs. (1.39) and (1.40) results in

$$\boxed{\nabla^2 \mathbf{A} - \mu\epsilon \frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu \mathbf{J}} \tag{1.42}$$

and

$$\boxed{\nabla^2 V - \mu\epsilon \frac{\partial^2 V}{\partial t^2} = -\frac{\rho_v}{\epsilon}} \tag{1.43}$$

which are inhomogeneous wave equations. Thus Maxwell's equations in terms of the potentials V and **A** reduce to the three equations (1.41) to (1.43). In other words, the three equations are equivalent to the ordinary form of Maxwell's equations in that potentials satisfying these equations always lead to a solution of Maxwell's equations for **E** and **B** when used with Eqs. (1.37) and (1.38). Integral solutions to Eqs. (1.42) and (1.43) are the so-called *retarded* potentials

$$\mathbf{A} = \int \frac{\mu[\mathbf{J}]\, dv}{4\pi R} \tag{1.44}$$

and

$$V = \int \frac{[\rho_v]\, dv}{4\pi\epsilon R} \tag{1.45}$$

where R is the distance from the source point to the field point, and the square brackets denote ρ_v and **J** are specified at a time $R(\mu\epsilon)^{1/2}$ earlier than for which **A** or V is being determined.

1.2.7 Time-harmonic Fields

Up to this point, we have considered the general case of arbitrary time variation of EM fields. In many practical situations, especially at low frequencies, it is sufficient to deal with only the steady-state (or equilibrium) solution of EM fields when produced by sinusoidal currents. Such fields are said to be sinusoidal time-varying or time-harmonic, that is, they vary at a sinusoidal frequency ω. An arbitrary time-dependent field $\mathbf{F}(x, y, z, t)$ or $\mathbf{F}(\mathbf{r}, t)$ can be expressed as

$$\boxed{\mathbf{F}(\mathbf{r}, t) = \mathrm{Re}\left[\mathbf{F}_s(\mathbf{r})e^{j\omega t}\right]} \tag{1.46}$$

where $\mathbf{F}_s(\mathbf{r}) = \mathbf{F}_s(x, y, z)$ is the phasor form of $\mathbf{F}(\mathbf{r}, t)$ and is in general complex, Re[] indicates "taking the real part of" quantity in brackets, and ω is the angular frequency (in rad/s) of the sinusoidal excitation. The EM field quantities can be represented in phasor notation as

$$\begin{bmatrix} \mathbf{E}(\mathbf{r}, t) \\ \mathbf{D}(\mathbf{r}, t) \\ \mathbf{H}(\mathbf{r}, t) \\ \mathbf{B}(\mathbf{r}, t) \end{bmatrix} = \begin{bmatrix} \mathbf{E}_s(\mathbf{r}) \\ \mathbf{D}_s(\mathbf{r}) \\ \mathbf{H}_s(\mathbf{r}) \\ \mathbf{B}_s(\mathbf{r}) \end{bmatrix} e^{j\omega t} \tag{1.47}$$

Using the phasor representation allows us to replace the time derivations $\partial/\partial t$ by $j\omega$ since

$$\frac{\partial e^{j\omega t}}{\partial t} = j\omega e^{j\omega t}$$

Thus Maxwell's equations, in sinusoidal steady state, become

$$\nabla \cdot \mathbf{D}_s = \rho_{vs} \tag{1.48a}$$

$$\nabla \cdot \mathbf{B}_s = 0 \tag{1.48b}$$

$$\nabla \times \mathbf{E}_s = -j\omega \mathbf{B}_s - \mathbf{J}_{ms} \tag{1.48c}$$

$$\nabla \times \mathbf{H}_s = \mathbf{J}_{es} + j\omega \mathbf{D}_s \tag{1.48d}$$

We should observe that the effect of the time-harmonic assumption is to eliminate the time dependence from Maxwell's equations, thereby reducing the time-space dependence to space dependence only. This simplification does not exclude more general time-varying fields if we consider ω to be one element of an entire frequency spectrum, with all the Fourier components superposed. In other words, a nonsinusoidal field can be represented as

$$\mathbf{F}(\mathbf{r}, t) = \mathrm{Re}\left[\int_{-\infty}^{\infty} \mathbf{F}_s(\mathbf{r}, \omega)e^{j\omega t}\, d\omega\right] \tag{1.49}$$

Thus the solutions to Maxwell's equations for a nonsinusoidal field can be obtained by summing all the Fourier components $\mathbf{F}_s(\mathbf{r}, \omega)$ over ω. Henceforth, we drop the subscript s denoting phasor quantity when no confusion results.

Replacing the time derivative in Eq. (1.34) by $(j\omega)^2$ yields the scalar wave equation in phasor representation as

$$\nabla^2 \Psi + k^2 \Psi = 0 \tag{1.50}$$

where k is the propagation constant (in *rad/m*), given by

$$k = \frac{\omega}{u} = \frac{2\pi f}{u} = \frac{2\pi}{\lambda} \tag{1.51}$$

We recall that Eqs. (1.31) to (1.34) were obtained assuming that $\rho_v = 0 = \mathbf{J}$. If $\rho_v \neq 0 \neq \mathbf{J}$, Eq. (1.50) will have the general form (see Prob. 1.4)

$$\boxed{\nabla^2 \Psi + k^2 \Psi = g} \tag{1.52}$$

We notice that this Helmholtz equation reduces to:

(1) Poisson's equation

$$\nabla^2 \Psi = g \tag{1.53}$$

when $k = 0$ (i.e., $\omega = 0$ for static case).

(2) Laplace's equation

$$\nabla^2 \Psi = 0 \tag{1.54}$$

when $k = 0 = g$.

Thus Poisson's and Laplace's equations are special cases of the Helmholtz equation. Note that function Ψ is said to be *harmonic* if it satisfies Laplace's equation.

Example 1.1

From the divergence theorem, derive Green's theorem

$$\int_v \left(U\nabla^2 V - V\nabla^2 U \right) dv = \oint_S \left(U\frac{\partial V}{\partial n} - V\frac{\partial U}{\partial n} \right) \cdot d\mathbf{S}$$

where $\frac{\partial \Phi}{\partial n} = \nabla \Phi \cdot \mathbf{a}_n$ is the directional derivation of Φ along the outward normal to S. □

Solution

In Eq. (1.1), let $\mathbf{F} = U\nabla V$, then

$$\int_v \nabla \cdot (U\nabla V)\, dv = \oint_S U\nabla V \cdot d\mathbf{S} \tag{1.55}$$

But

$$\begin{aligned} \nabla \cdot (U\nabla V) &= U\nabla \cdot \nabla V + \nabla V \cdot \nabla U \\ &= U\nabla^2 V + \nabla U \cdot \nabla V \end{aligned}$$

Substituting this into Eq. (1.55) gives *Green's first identity:*

$$\int_v \left(U\nabla^2 V + \nabla U \cdot \nabla V\right) dv = \oint_S U\nabla V \cdot d\mathbf{S} \tag{1.56}$$

By interchanging U and V in Eq. (1.56), we obtain

$$\int_v \left(V\nabla^2 U + \nabla V \cdot \nabla U\right) dv = \oint_S V\nabla U \cdot d\mathbf{S} \tag{1.57}$$

Subtracting Eq. (1.57) from Eq. (1.56) leads to *Green's second identity* or Green's theorem:

$$\int_v \left(U\nabla^2 V - V\nabla^2 U\right) dv = \oint_S (U\nabla V - V\nabla U) \cdot d\mathbf{S} \quad \blacksquare$$

Example 1.2

Show that the continuity equation is implicit (or incorporated) in Maxwell's equations.

Solution

According to Eq. (1.36), the divergence of the curl of any vector field is zero. Hence, taking the divergence of Eq. (1.22d) gives

$$0 = \nabla \cdot \nabla \times \mathbf{H} = \nabla \cdot \mathbf{J} + \frac{\partial}{\partial t}\nabla \cdot \mathbf{D}$$

But $\nabla \cdot \mathbf{D} = \rho_v$ from Eq. (1.22a). Thus,

$$0 = \nabla \cdot \mathbf{J} + \frac{\partial \rho_v}{\partial t}$$

which is the continuity equation. $\blacksquare$

Example 1.3

Express:

(a) $\mathbf{E} = 10\sin(\omega t - kz)\mathbf{a}_x + 20\cos(\omega t - kz)\mathbf{a}_y$ in phasor form.

(b) $\mathbf{H}_s = (4 - j3)\sin x\mathbf{a}_x + \frac{e^{j10^\circ}}{x}\mathbf{a}_z$ in instantaneous form.

Solution

(a) We can express $\sin\theta$ as $\cos(\theta - \pi/2)$. Hence,

$$\begin{aligned}\mathbf{E} &= 10\cos(\omega t - kz - \pi/2)\mathbf{a}_x + 20\cos(\omega t - kz)\mathbf{a}_y \\ &= \mathrm{Re}\left[\left(10e^{-jkz}e^{-j\pi/2}\mathbf{a}_x + 20e^{-jkz}\mathbf{a}_y\right)e^{j\omega t}\right] \\ &= \mathrm{Re}\left[\mathbf{E}_s e^{j\omega t}\right]\end{aligned}$$

Thus,

$$\begin{aligned}\mathbf{E}_s &= 10e^{-jkz}e^{-j\pi/2}\mathbf{a}_x + 20e^{-jkz}\mathbf{a}_y \\ &= \left(-j10\mathbf{a}_x + 20\mathbf{a}_y\right)e^{-jkz}\end{aligned}$$

(b) Since

$$\begin{aligned}\mathbf{H} &= \mathrm{Re}\left[\mathbf{H}_s e^{j\omega t}\right] \\ &= \mathrm{Re}\left[5\sin x e^{j(\omega t - 36.87^\circ)}\mathbf{a}_x + \frac{1}{x}e^{j(\omega t + 10^\circ)}\mathbf{a}_z\right] \\ &= \left[5\sin x\cos(\omega t - 36.87^\circ)\mathbf{a}_x + \frac{1}{x}\cos(\omega t + 10^\circ)\mathbf{a}_z\right]\end{aligned}$$ ▮

1.3 Classification of EM Problems

Classifying EM problems will help us later to answer the question of what method is best for solving a given problem. Continuum problems are categorized differently depending on the particular item of interest, which could be one of these:

(1) the solution region of the problem,

(2) the nature of the equation describing the problem, or

(3) the associated boundary conditions.

(In fact, the above three items define a problem uniquely.) It will soon become evident that these classifications are sometimes not independent of each other.

1.3.1 Classification of Solution Regions

In terms of the solution region or problem domain, the problem could be an interior problem, also variably called an inner, closed, or bounded problem, or an exterior problem, also variably called an outer, open, or unbounded problem.

Consider the solution region R with boundary S, as shown in Fig. 1.2. If part or all of S is at infinity, R is exterior/open, otherwise R is interior/closed. For example, wave propagation in a waveguide is an interior problem, whereas while wave propagation in free space — scattering of EM waves by raindrops, and radiation from a dipole antenna — are exterior problems.

A problem can also be classified in terms of the electrical, constitutive properties (σ, ϵ, μ) of the solution region. As mentioned in Section 1.2.4, the solution region could be linear (or nonlinear), homogeneous (or inhomogeneous), and isotropic (or anisotropic). We shall be concerned, for the most part, with linear, homogeneous, isotropic media in this text.

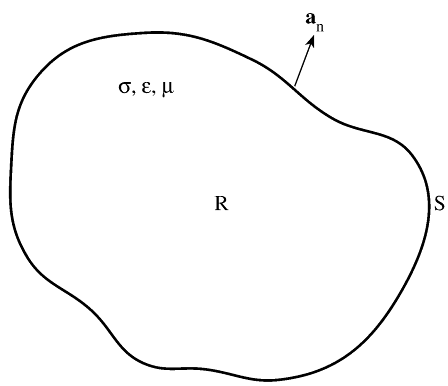

Figure 1.2
Solution region R with boundary S.

1.3.2 Classification of Differential Equations

EM problems are classified in terms of the equations describing them. The equations could be differential or integral or both. Most EM problems can be stated in terms of an operator equation

$$\boxed{L\Phi = g} \tag{1.58}$$

where L is an operator (differential, integral, or integro-differential), g is the known excitation or source, and Φ is the unknown function to be determined. A typical example is the electrostatic problem involving Poisson's equation. In differential form, Eq. (1.58) becomes

$$-\nabla^2 V = \frac{\rho_v}{\epsilon} \tag{1.59}$$

so that $L = -\nabla^2$ is the Laplacian operator, $g = \rho_v/\epsilon$ is the source term, and $\Phi = V$ is the electric potential. In integral form, Poisson's equation is of the form

$$V = \int \frac{\rho_v dv}{4\pi\epsilon r^2} \tag{1.60}$$

so that

$$L = \int \frac{dv}{4\pi r^2}, \quad g = V, \quad \text{and} \quad \Phi = \rho_v/\epsilon$$

In this section, we shall limit our discussion to differential equations; integral equations will be considered in detail in Chapter 5.

As observed in Eqs. (1.52) to (1.54), EM problems involve linear, second-order differential equations. In general, a second-order partial differential equation (PDE) is given by

$$a\frac{\partial^2\Phi}{\partial x^2} + b\frac{\partial^2\Phi}{\partial x\partial y} + c\frac{\partial^2\Phi}{\partial y^2} + d\frac{\partial\Phi}{\partial x} + e\frac{\partial\Phi}{\partial y} + f\Phi = g$$

or simply

$$\boxed{a\Phi_{xx} + b\Phi_{xy} + c\Phi_{yy} + d\Phi_x + e\Phi_y + f\Phi = g} \tag{1.61}$$

The coefficients, a, b and c in general are functions of x and y; they may also depend on Φ itself, in which case the PDE is said to be *nonlinear*. A PDE in which $g(x, y)$ in Eq. (1.61) equals zero is termed *homogeneous;* it is *inhomogeneous* if $g(x, y) \neq 0$. Notice that Eq. (1.61) has the same form as Eq. (1.58), where L is now a differential operator given by

$$L = a\frac{\partial^2}{\partial x^2} + b\frac{\partial^2}{\partial x \partial y} + c\frac{\partial^2}{\partial y^2} + d\frac{\partial}{\partial x} + e\frac{\partial}{\partial y} + f \tag{1.62}$$

A PDE in general can have both boundary values and initial values. PDEs whose boundary conditions are specified are called *steady-state equations*. If only initial values are specified, they are called *transient equations*.

Any linear second-order PDE can be classified as elliptic, hyperbolic, or parabolic depending on the coefficients a, b, and c. Equation (1.61) is said to be:

$$\boxed{\begin{array}{rl} \text{elliptic if} & b^2 - 4ac < 0 \\ \text{hyperbolic if} & b^2 - 4ac > 0 \\ \text{parabolic if} & b^2 - 4ac = 0 \end{array}} \tag{1.63}$$

The terms *hyperbolic, parabolic,* and *elliptic* are derived from the fact that the quadratic equation

$$ax^2 + bxy + cy^2 + dx + ey + f = 0$$

represents a hyperbola, parabola, or ellipse if $b^2 - 4ac$ is positive, zero, or negative, respectively. In each of these categories, there are PDEs that model certain physical phenomena. Such phenomena are not limited to EM but extend to almost all areas of science and engineering. Thus the mathematical model specified in Eq. (1.61) arises in problems involving heat transfer, boundary-layer flow, vibrations, elasticity, electrostatic, wave propagation, and so on.

Elliptic PDEs are associated with steady-state phenomena, i.e., boundary-value problems. Typical examples of this type of PDE include Laplace's equation

$$\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} = 0 \tag{1.64}$$

and Poisson's equation

$$\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} = g(x, y) \tag{1.65}$$

where in both cases $a = c = 1, b = 0$. An elliptic PDE usually models an interior problem, and hence the solution region is usually closed or bounded as in Fig. 1.3 (a).

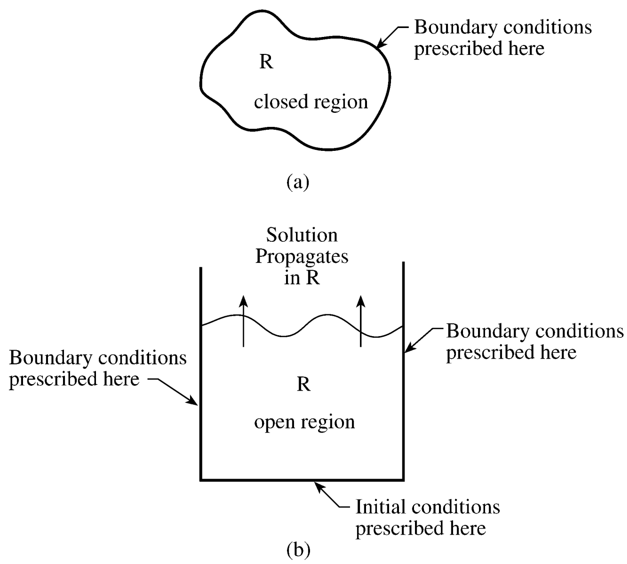

Figure 1.3
(a) Elliptic, (b) parabolic, or hyperbolic problem.

Hyperbolic PDEs arise in propagation problems. The solution region is usually open so that a solution advances outward indefinitely from initial conditions while always satisfying specified boundary conditions. A typical example of hyperbolic PDE is the wave equation in one dimension

$$\frac{\partial^2 \Phi}{\partial x^2} = \frac{1}{u^2} \frac{\partial^2 \Phi}{\partial t^2} \tag{1.66}$$

where $a = u^2, b = 0, c = -1$. Notice that the wave equation in (1.50) is not hyperbolic but elliptic, since the time-dependence has been suppressed and the equation is merely the steady-state solution of Eq. (1.34).

Parabolic PDEs are generally associated with problems in which the quantity of interest varies slowly in comparison with the random motions which produce the variations. The most common parabolic PDE is the diffusion (or heat) equation in one dimension

$$\frac{\partial^2 \Phi}{\partial x^2} = k \frac{\partial \Phi}{\partial t} \tag{1.67}$$

where $a = 1, b = 0 = c$. Like hyperbolic PDE, the solution region for parabolic PDE is usually open, as in Fig. 1.3 (b). The initial and boundary conditions typically associated with parabolic equations resemble those for hyperbolic problems except that only one initial condition at $t = 0$ is necessary since Eq. (1.67) is only first order in time. Also, parabolic and hyperbolic equations are solved using similar techniques, whereas elliptic equations are usually more difficult and require different techniques.

Note that: (1) since the coefficients a, b, and c are in general functions of x and y, the classification of Eq. (1.61) may change from point to point in the solution region, and (2) PDEs with more than two independent variables $(x, y, z, t, \dots)$ may not fit as neatly into the classification above. A summary of our discussion so far in this section is shown in Table 1.1.

Table 1.1 Classification of Partial Differential Equations

Type	Sign of $b^2 - 4ac$	Example	Solution region
Elliptic	$-$	Laplace's equation: $\Phi_{xx} + \Phi_{yy} = 0$	Closed
Hyperbolic	$+$	Wave equation: $u^2\Phi_{xx} = \Phi_{tt}$	Open
Parabolic	0	Diffusion equation: $\Phi_{xx} = k\Phi_t$	Open

The type of problem represented by Eq. (1.58) is said to be *deterministic*, since the quantity of interest can be determined directly. Another type of problem where the quantity is found indirectly is called *nondeterministic* or *eigenvalue*. The *standard eigenproblem* is of the form

$$L\Phi = \lambda\Phi \tag{1.68}$$

where the source term in Eq. (1.58) has been replaced by $\lambda\Phi$. A more general version is the *generalized eigenproblem* having the form

$$\boxed{L\Phi = \lambda M\Phi} \tag{1.69}$$

where M, like L, is a linear operator for EM problems. In Eqs. (1.68) and (1.69), only some particular values of λ called *eigenvalues* are permissible; associated with these values are the corresponding solutions Φ called *eigenfunctions*. Eigenproblems are usually encountered in vibration and waveguide problems where the eigenvalues λ correspond to physical quantities such as resonance and cutoff frequencies, respectively.

1.3.3 Classification of Boundary Conditions

Our problem consists of finding the unknown function Φ of a partial differential equation. In addition to the fact that Φ satisfies Eq. (1.58) within a prescribed solution region R, Φ must satisfy certain conditions on S, the boundary of R. Usually these boundary conditions are of the Dirichlet and Neumann types. Where a boundary has both, a mixed boundary condition is said to exist.

(1) Dirichlet boundary condition:

$$\Phi(\mathbf{r}) = 0, \qquad \mathbf{r} \quad \text{on S} \tag{1.70}$$

(2) Neumann boundary condition:

$$\frac{\partial \Phi(\mathbf{r})}{\partial n} = 0, \qquad \mathbf{r} \quad \text{on S} \;, \tag{1.71}$$

i.e., the normal derivative of Φ vanishes on S.

(3) Mixed boundary condition:

$$\frac{\partial \Phi(\mathbf{r})}{\partial n} + h(\mathbf{r})\Phi(\mathbf{r}) = 0, \qquad \mathbf{r} \quad \text{on S} \;, \tag{1.72}$$

where $h(\mathbf{r})$ is a known function and $\frac{\partial \Phi}{\partial n}$ is the directional derivative of Φ along the outward normal to the boundary S, i.e.,

$$\frac{\partial \Phi}{\partial n} = \nabla \Phi \cdot \mathbf{a}_n \tag{1.73}$$

where $\mathbf{a}_n$ is a unit normal directed out of R, as shown in Fig. 1.2. Note that the Neumann boundary condition is a special case of the mixed condition with $h(\mathbf{r}) = 0$.

The conditions in Eq. (1.70) to (1.72) are called *homogeneous boundary conditions*. The more general ones are the inhomogeneous:

Dirichlet:

$$\Phi(\mathbf{r}) = p(\mathbf{r}), \qquad \mathbf{r} \quad \text{on S} \tag{1.74}$$

Neumann:

$$\frac{\partial \Phi(\mathbf{r})}{\partial n} = q(\mathbf{r}), \quad \mathbf{r} \;\; \text{on} \;\; \mathbf{S} \tag{1.75}$$

Mixed:

$$\frac{\partial \Phi(\mathbf{r})}{\partial n} + h(\mathbf{r})\Phi(\mathbf{r}) = w(\mathbf{r}), \quad \mathbf{r} \;\; \text{on} \;\; \mathbf{S} \tag{1.76}$$

where $p(\mathbf{r})$, $q(\mathbf{r})$, and $w(\mathbf{r})$ are explicitly known functions on the boundary **S**. For example, $\Phi(0) = 1$ is an inhomogeneous Dirichlet boundary condition, and the associated homogeneous counterpart is $\Phi(0) = 0$. Also $\Phi'(1) = 2$ and $\Phi'(1) = 0$ are, respectively, inhomogeneous and homogeneous Neumann boundary conditions. In electrostatics, for example, if the value of electric potential is specified on **S**, we have Dirichlet boundary condition, whereas if the surface charge $(\rho_s = D_n = \epsilon \frac{\partial V}{\partial n})$ is specified, the boundary condition is Neumann. The problem of finding a function Φ that is harmonic in a region is called *Dirichlet problem* (or *Neumann problem*) if Φ (or $\frac{\partial \Phi}{\partial n}$) is prescribed on the boundary of the region.

It is worth observing that the term "homogeneous" has been used to mean different things. The solution region could be homogeneous meaning that σ, ϵ, and μ are constant within R; the PDE could be homogeneous if $g = 0$ so that $L\Phi = 0$; and the boundary conditions are homogeneous when $p(\mathbf{r}) = q(\mathbf{r}) = w(\mathbf{r}) = 0$.

Example 1.4

Classify these equations as elliptic, hyperbolic, or parabolic:

(a) $4\Phi_{xx} + 2\Phi_x + \Phi_y + x + y = 0$

(b) $e^x \dfrac{\partial^2 V}{\partial x^2} + \cos y \dfrac{\partial^2 V}{\partial x \partial y} - \dfrac{\partial^2 V}{\partial y^2} = 0$.

State whether the equations are homogeneous or inhomogeneous. □

Solution

(a) In this PDE, $a = 4$, $b = 0 = c$. Hence

$$b^2 - 4ac = 0 ,$$

i.e., the PDE is parabolic. Since $g = -x - y$, the PDE is inhomogeneous.

(b) For this PDE, $a = e^x$, $b = \cos y$, $c = -1$. Hence

$$b^2 - 4ac = \cos^2 y + 4e^x > 0$$

and the PDE is hyperbolic. Since $g = 0$, the PDE is homogeneous. ∎

1.4 Some Important Theorems

Two theorems are of fundamental importance in solving EM problems. These are the principle of superposition and the uniqueness theorem.

1.4.1 Superposition Principle

The principle of superposition is applied in several ways. We shall consider two of these.

If each member of a set of functions Φ_n, $n = 1, 2, \ldots, N$, is a solution to the PDE $L\Phi = 0$ with some prescribed boundary conditions, then a linear combination

$$\Phi_N = \Phi_0 + \sum_{n=1}^{N} a_n \Phi_n \tag{1.77}$$

also satisfies $L\Phi = g$.

Given a problem described by the PDE

$$L\Phi = g \tag{1.78}$$

subject to the boundary conditions

$$\begin{aligned} M_1(s) &= h_1 \\ M_2(s) &= h_2 \\ &\vdots \\ M_N(s) &= h_N \ , \end{aligned} \tag{1.79}$$

as long as L is linear, we may divide the problem into a series of problems as follows:

$$\begin{array}{cccc} L\Phi_0 = g & L\Phi_1 = 0 & \cdots & L\Phi_N = 0 \\ M_1(s) = 0 & M_1(s) = h_1 & \cdots & M_1(s) = 0 \\ M_2(s) = 0 & M_2(s) = 0 & \cdots & M_2(s) = 0 \\ \vdots & \vdots & & \vdots \\ M_N(s) = 0 & M_N(s) = 0 & \cdots & M_N(s) = h_N \end{array} \tag{1.80}$$

where $\Phi_0, \Phi_1, \ldots, \Phi_N$ are the solutions to the reduced problems, which are easier to solve than the original problem. The solution to the original problem is given by

$$\Phi = \sum_{n=0}^{N} \Phi_n \tag{1.81}$$

1.4.2 Uniqueness Theorem

This theorem guarantees that the solution obtained for a PDE with some prescribed boundary conditions is the only one possible. For EM problems, the theorem may be stated as follows: If in any way a set of fields $(\mathbf{E}, \mathbf{H})$ is found which satisfies simultaneously Maxwell's equations and the prescribed boundary conditions, this set is unique. Therefore, a field is uniquely specified by the sources $(\rho_v, \mathbf{J})$ within the medium plus the tangential components of $\mathbf{E}$ or $\mathbf{H}$ over the boundary.

To prove the uniqueness theorem, suppose there exist two solutions (with subscripts 1 and 2) that satisfy Maxwell's equations

$$\nabla \cdot \epsilon \mathbf{E}_{1,2} = \rho_v \tag{1.82a}$$

$$\nabla \cdot \mathbf{H}_{1,2} = 0 \tag{1.82b}$$

$$\nabla \times \mathbf{E}_{1,2} = -\mu \frac{\partial \mathbf{H}_{1,2}}{\partial t} \tag{1.82c}$$

$$\nabla \times \mathbf{H}_{1,2} = \mathbf{J} + \sigma \mathbf{E}_{1,2} + \epsilon \frac{\partial \mathbf{E}_{1,2}}{\partial t} \tag{1.82d}$$

If we denote the difference of the two fields as $\Delta \mathbf{E} = \mathbf{E}_2 - \mathbf{E}_1$ and $\Delta \mathbf{H} = \mathbf{H}_2 - \mathbf{H}_1$,

$\Delta \mathbf{E}$ and $\Delta \mathbf{H}$ must satisfy the source-free Maxwell's equations, i.e.,

$$\nabla \cdot \epsilon \Delta \mathbf{E} = 0 \tag{1.83a}$$

$$\nabla \cdot \Delta \mathbf{H} = 0 \tag{1.83b}$$

$$\nabla \times \Delta \mathbf{E} = -\mu \frac{\partial \Delta \mathbf{H}}{\partial t} \tag{1.83c}$$

$$\nabla \times \Delta \mathbf{H} = \sigma \Delta \mathbf{E} + \epsilon \frac{\partial \Delta \mathbf{E}}{\partial t} \tag{1.83d}$$

Dotting both sides of Eq. (1.83d) with $\Delta \mathbf{E}$ gives

$$\Delta \mathbf{E} \cdot \nabla \times \Delta \mathbf{H} = \sigma |\Delta \mathbf{E}|^2 + \epsilon \nabla \mathbf{E} \cdot \frac{\partial \Delta \mathbf{E}}{\partial t} \tag{1.84}$$

Using the vector identity

$$\mathbf{A} \cdot (\nabla \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \nabla \cdot (\mathbf{A} \times \mathbf{B})$$

and Eq. (1.83c), Eq. (1.84) becomes

$$\nabla \cdot (\Delta \mathbf{E} \times \Delta \mathbf{H}) = -\frac{1}{2} \frac{\partial}{\partial t} \left(\mu |\Delta \mathbf{H}|^2 + \epsilon |\Delta \mathbf{E}|^2 \right) - \sigma |\Delta \mathbf{E}|^2$$

Integrating over volume v bounded by surface $\mathbf{S}$ and applying divergence theorem to the left-hand side, we obtain

$$\oint_S (\Delta \mathbf{E} \times \Delta \mathbf{H}) \cdot d\mathbf{S} = -\frac{\partial}{\partial t} \int_v \left[\frac{1}{2} \epsilon |\Delta \mathbf{E}|^2 + \frac{1}{2} \mu |\Delta \mathbf{H}|^2 \right] dv - \int_v \sigma |\Delta \mathbf{E}| \, dv \tag{1.85}$$

showing that $\Delta \mathbf{E}$ and $\Delta \mathbf{H}$ satisfy the Poynting theorem just as $\mathbf{E}_{1,2}$ and $\mathbf{H}_{1,2}$. Only the tangential components of $\Delta \mathbf{E}$ and $\Delta \mathbf{H}$ contribute to the surface integral on the left side of Eq. (1.85). Therefore, if the tangential components of $\mathbf{E}_1$ and $\mathbf{E}_2$ or $\mathbf{H}_1$ and $\mathbf{H}_2$ are equal over $\mathbf{S}$ (thereby satisfying Eq. (1.27)), the tangential components of $\Delta \mathbf{E}$ and $\Delta \mathbf{H}$ vanish on $\mathbf{S}$. Consequently, the surface integral in Eq. (1.85) is identically zero, and hence the right side of the equation must vanish also. It follows that $\Delta \mathbf{E} = 0$ due to the second integral on the right side and hence also $\Delta \mathbf{H} = 0$ throughout the volume. Thus $\mathbf{E}_1 = \mathbf{E}_2$ and $\mathbf{H}_1 = \mathbf{H}_2$, confirming that the solution is unique.

The theorem just proved for time-varying fields also holds for static fields as a special case. In terms of electrostatic potential V, the uniqueness theorem may be stated as follows: A solution to $\nabla^2 V = 0$ is uniquely determined by specifying either the value of V or the normal component of ∇V at each point on the boundary surface. For a magnetostatic field, the theorem becomes: A solution of $\nabla^2 \mathbf{A} = 0$ (and $\nabla \cdot \mathbf{A} = 0$) is uniquely determined by specifying the value of $\mathbf{A}$ or the tangential component of $\mathbf{B} = (\nabla \times \mathbf{A})$ at each point on the boundary surface.

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Problems

1.1 In a coordinate system of your choice, prove that:

(a) $\nabla \times \nabla\Phi = 0$,

(b) $\nabla \cdot \nabla \times \mathbf{F} = 0$,

(c) $\nabla \times \nabla \times \mathbf{F} = \nabla(\nabla \cdot \mathbf{F}) - \nabla^2\mathbf{F}$,

where Φ and $\mathbf{F}$ are scalar and vector fields, respectively.

1.2 If U and V are scalar fields, show that

$$\oint_L U\nabla V \cdot d\mathbf{l} = -\oint_L V\nabla U \cdot d\mathbf{l}$$

1.3 Show that in a source-free region ($\mathbf{J} = 0$, $\rho_v = 0$), Maxwell's equations can be reduced to the two curl equations.

1.4 In deriving the wave equations (1.31) and (1.32), we assumed a source-free medium ($\mathbf{J} = 0$, $\rho_v = 0$). Show that if $\rho_v \neq 0$, $\mathbf{J} \neq 0$, the equations become

$$\nabla^2\mathbf{E} - \frac{1}{c^2}\frac{\partial^2\mathbf{E}}{\partial t^2} = \nabla(\rho_v/\epsilon) + \mu\frac{\partial\mathbf{J}}{\partial t} \,,$$

$$\nabla^2\mathbf{H} - \frac{1}{c^2}\frac{\partial^2\mathbf{H}}{\partial t^2} = -\nabla \times \mathbf{J}$$

What assumptions have you made to arrive at these expressions?

1.5 Determine whether the fields

$$\mathbf{E} = 20\sin(\omega t - kz)\mathbf{a}_x - 10\cos(\omega t + kz)\mathbf{a}_y$$
$$\mathbf{H} = \frac{k}{\omega\mu_o}\left[-10\cos(\omega t + kz)\mathbf{a}_x + 20\sin(\omega t - kz)\mathbf{a}_y\right] ,$$

where $k = \omega\sqrt{\mu_o\epsilon_o}$, satisfy Maxwell's equations.

1.6 In free space, the electric flux density is given by

$$\mathbf{D} = D_0 \cos(\omega t + \beta z)\mathbf{a}_x$$

Use Maxwell's equation to find **H**.

1.7 In free space, a source radiates the magnetic field

$$\mathbf{H}_s = H_0 \frac{e^{-j\beta\rho}}{\sqrt{\rho}}\mathbf{a}_\phi$$

where $\beta = \omega\sqrt{\mu_0\epsilon_0}$. Determine $\mathbf{E}_s$.

1.8 An electric dipole of length L in free space has a radical field given in spherical system (r, θ, ϕ) as

$$\mathbf{H}_s = \frac{IL}{4\pi r}\sin\theta\left(\frac{1}{r} + j\beta\right)e^{-j\beta r}\mathbf{a}_\phi$$

Find $\mathbf{E}_s$ using Maxwell's equations.

1.9 Show that the electric field

$$\mathbf{E}_s = 20\sin(k_x x)\cos(k_y y)\mathbf{a}_z ,$$

where $k_x^2 + k_y^2 = \omega^2\mu_0\epsilon_0$, can be represented as the superposition of four propagating plane waves. Find the corresponding $\mathbf{H}_s$ field.

1.10 (a) Express $I_s = e^{-jz}\sin\pi x\cos\pi y$ in instantaneous form.

(b) Determine the phasor form of $V = 20\sin(\omega t - 2x) - 10\cos(\omega t - 4x)$

1.11 For each of the following phasors, determine the corresponding instantaneous form:

(a) $\mathbf{A}_s = (\mathbf{a}_x + j\mathbf{a}_y)e^{-2jz}$

(b) $\mathbf{B}_s = j10\sin x\mathbf{a}_x + 5e^{-j12z-\pi/4}\mathbf{a}_z$

(c) $\mathbf{C}_s = \frac{2}{j}e^{-j3x}\cos 2x + e^{3x-j4x}$

1.12 Show that a time-harmonic EM field in a conducting medium $(\sigma >> \omega\epsilon)$ satisfies the diffusion equation

$$\nabla^2\mathbf{E}_s - j\omega\mu\sigma\mathbf{E}_s = 0$$

1.13 Show that in an inhomogeneous medium, the wave equations become

$$\nabla \times \left(\frac{1}{j\omega\mu}\nabla \times \mathbf{E}_s\right) + j\omega\epsilon\mathbf{E}_s = 0 \,,$$
$$\nabla \times \left(\frac{1}{j\omega\epsilon}\nabla \times \mathbf{H}_s\right) + j\omega\mu\mathbf{H}_s = 0$$

1.14 Show that the time-harmonic potential function V_s and $\mathbf{A}_s$ satisfy the following inhomogeneous wave equation

$$\nabla^2 V_s + k^2 V_s = -\frac{\rho_{vs}}{\epsilon}$$
$$\nabla^2 \mathbf{A}_s + k^2 \mathbf{A}_s = -\mu \mathbf{J}_s$$

where $k^2 = \omega^2\mu\epsilon$.

1.15 Classify the following PDEs as elliptic, parabolic, or hyperbolic.

(a) $\Phi_{xx} + 2\Phi_{xy} + 5\Phi_{yy} = 0$

(b) $(y^2 + 1)\Phi_{xx} + (x^2 + 1)\Phi_{yy} = 0$

(c) $\Phi_{xx} - 2\cos x\,\Phi_{xy} - (3 + \sin^2 x)\Phi_{yy} - y\Phi_y = 0$

(d) $x^2\Phi_{xx} - 2xy\Phi_{xy} + y^2\Phi_{yy} + x\Phi_x + y\Phi_y = 0$

1.16 Repeat Prob. 1.15 for the following PDEs:

(a) $\alpha\dfrac{\partial^2\Phi}{\partial x^2} = \beta\dfrac{\partial\Phi}{\partial x} + \dfrac{\partial\Phi}{\partial t}$ $\qquad (\alpha, \beta = \text{constant})$
which is called convective heat equation.

(b) $\nabla^2\phi + \lambda\Phi = 0$
which is the Helmholtz equation.

(c) $\nabla^2\Phi + [\lambda - \rho(x)]\Phi = 0$
which is the time-independent Schrodinger equation.