Chapter 4

Variational Methods

"You can do anything if you have enthusiasm. Enthusiasm is the yeast that makes your hopes rise to the stars. Enthusiasm is the spark in your eye, the swing in your gait, the grip of your hand, the irresistible surge of your will and your energy to execute your ideas. Enthusiasts are fighters; they have fortitude; they have staying qualities. Enthusiasm is at the bottom of all progress! With it, there is accomplishment. Without it, there are only alibis." Henry Ford

4.1 Introduction

In solving problems arising from mathematical physics and engineering, we find that it is often possible to replace the problem of integrating a differential equation by the equivalent problem of seeking a function that gives a minimum value of some integral. Problems of this type are called *variational problems*. The methods that allow us to reduce the problem of integrating a differential equation to the equivalent variational problem are usually called *variational methods* [1]. The variational methods form a common base for both the method of moments (MOM) and the finite element method (FEM). Therefore, it is appropriate that we study the variational methods before MOM and FEM. Besides, it is relatively easy to formulate the solution of certain differential and integral equations in variational terms. Also, variational methods give accurate results without making excessive demands on computer storage and time.

Variational methods can be classified into two groups: direct and indirect methods. The direct method is the classical Rayleigh-Ritz method, while the indirect methods are collectively referred to as the method of weighted residuals: collocation (or point-matching), subdomain, Galerkin, and least square methods. The variational solution of a given PDE using an indirect method usually involves two basic steps [2]:

- cast the PDE into variational form, and
- determine the approximate solution using one of the methods.

The literature on the theory and applications of variational methods to EM problems is quite extensive, and no attempt will be made to provide an exhaustive list of references. Numerous additional references may be found in those cited in this chapter. Owing to a lack of space, we only can hint at some of the topics usually covered in an introduction to this subject.

4.2 Operators in Linear Spaces

In this section, we will review some principles of operators in linear spaces and establish notation [2]–[5]. We define the *inner (dot* or *scalar) product* of functions u and v as

$$\boxed{\langle u, v\rangle = \int_\Omega u v^* \, d\Omega} \tag{4.1}$$

where * denotes the complex conjugate and the integration is performed over Ω, which may be one-, two-, or three-dimensional physical space depending on the problem. In a sense, the inner product $\langle u, v\rangle$ gives the component or projection of function u in the direction of v. If $\mathbf{u}$ and $\mathbf{v}$ are vector fields, we modify Eq. (4.1) slightly to include a dot between them, i.e.,

$$\langle \mathbf{u}, \mathbf{v}\rangle = \int_\Omega \mathbf{u} \cdot \mathbf{v}^* \, d\Omega \tag{4.2}$$

However, we shall consider u and v to be complex-valued scalar functions. For each pair of u and v belonging to the linear space, a number $\langle u, v\rangle$ is obtained that satisfies:

$$\boxed{\begin{aligned}
&(1) \quad \langle u, v\rangle = \langle v, u\rangle^* , && (4.3a)\\
&(2) \quad \langle \alpha u_1 + \beta u_2, v\rangle = \alpha\langle u_1, v\rangle + \beta\langle u_2, v\rangle , && (4.3b)\\
&(3) \quad \langle u, v\rangle > 0 \quad \text{if} \quad u \neq 0 , && (4.3c)\\
&(4) \quad \langle u, v\rangle = 0 \quad \text{if} \quad u = 0 && (4.3d)
\end{aligned}}$$

If $\langle u, v\rangle = 0$, u and v are said to be *orthogonal*. Notice that these properties mimic familiar properties of the dot product in three-dimensional space. Equation (4.3) is easily derived from Eq. (4.1). Note that from Eqs. (4.3a) and (4.3b),

$$\langle u, \alpha v\rangle = \alpha^*\langle v, u\rangle^* = \alpha^*\langle u, v\rangle$$

where α is a complex scalar.

Equation (4.1) is called an *unweighted* or *standard inner product*. A *weighted inner product* is given by

$$\langle u, v\rangle = \int_{\Omega} u v^* w \, d\Omega \tag{4.4}$$

where w is a suitable weight function.

We define the norm of the function u as

$$\|u\| = \sqrt{\langle u, u\rangle} \tag{4.5}$$

The norm is a measure of the "length" or "magnitude" of the function. (As far as a field is concerned, the norm is its rms value.) A vector is said to be *normal* if its norm is 1. Since the *Schwarz inequality*

$$|\langle u, v\rangle| \leq \|u\|\|v\| \tag{4.6}$$

holds for any inner product space, the angle θ between two nonzero vectors $\mathbf{u}$ and $\mathbf{v}$ can be obtained as

$$\theta = \cos^{-1} \frac{\langle \mathbf{u}, \mathbf{v}\rangle}{\|\mathbf{u}\|\|\mathbf{v}\|} \tag{4.7}$$

We now consider the operator equation

$$\boxed{L\Phi = g} \tag{4.8}$$

where L is any linear operator, Φ is the unknown function, and g is the source function. The space spanned by all functions resulting from the operator L is

$$\langle L\Phi, g\rangle = \langle \Phi, L^a g\rangle \tag{4.9}$$

The operator L is said to be:

(1) self-adjoint if $L = L^a$, i.e, $\langle L\Phi, g\rangle = \langle \Phi, Lg\rangle$,

(2) positive definite if $\langle L\Phi, \Phi\rangle > 0$ for any $\Phi \neq 0$ in the domain of L,

(3) negative definite if $\langle L\Phi, \Phi\rangle < 0$ for any $\Phi \neq 0$ in the domain of L.

The properties of the solution of Eq. (4.8) depend strongly on the properties of the operator L. If, for example, L is positive definite, we can easily show that the solution of Φ in Eq. (4.8) is unique, i.e., Eq. (4.8) cannot have more than one solution. To do this, suppose that Φ and Ψ are two solutions to Eq. (4.8) such that $L\Phi = g$ and $L\Psi = g$. Then, by virtue of linearity of L, $f = \Phi - \Psi$ is also a solution. Therefore, $Lf = 0$. Since L is positive definite, $f = 0$ implying that $\Phi = \Psi$ and confirming the uniqueness of the solution Φ.

Example 4.1

Find the inner product of $u(x) = 1 - x$ and $v(x) = 2x$ in the interval (0, 1). □

Solution
In this case, both u and v are real functions. Hence

$$\langle u, v\rangle = \langle v, u\rangle = \int_0^1 (1-x)2x\,dx$$
$$= 2\left(\frac{x^2}{2} - \frac{x^3}{3}\right)\Big|_0^1 = 0.333 \quad \blacksquare$$

Example 4.2
Show that the operator

$$L = -\nabla^2 = -\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}$$

is self-adjoint. □

Solution

$$\langle Lu, v\rangle = -\int_S v\nabla^2 u\,dS$$

Taking u and v to be real functions (for convenience) and applying the Green's identity

$$\oint_\ell v\frac{\partial u}{\partial n}\,dl = \int_S \nabla u\cdot\nabla v\,dS + \int_S v\nabla^2 u\,dS$$

yields

$$\langle Lu, v\rangle = \int_S \nabla u\cdot\nabla v\,dS - \oint_\ell v\frac{\partial u}{\partial n}\,dl \tag{4.10}$$

where S is bounded by ℓ and $\mathbf{n}$ is the outward normal. Similarly

$$\langle u, Lv\rangle = \int_S \nabla u\cdot\nabla v\,dS - \oint_\ell u\frac{\partial v}{\partial n}\,dl \tag{4.11}$$

The line integrals in Eqs. (4.10) and (4.11) vanish under either the homogeneous Dirichlet or Neumann boundary conditions. Under the homogeneous mixed boundary conditions, they become equal. Thus, L is self-adjoint under any one of these boundary conditions. L is also positive definite. ■

4.3 Calculus of Variations

The calculus of variations, an extension of ordinary calculus, is a discipline that is concerned primarily with the theory of maxima and minima. Here we are concerned

with seeking the extremum (minima or maxima) of an integral expression involving a function of functions or *functionals*. Whereas a function produces a number as a result of giving values to one or more independent variables, a functional produces a number that depends on the entire form of one or more functions between prescribed limits. In a sense, a functional is a measure of the function. A simple example is the inner product $\langle u, v \rangle$.

In the calculus of variation, we are interested in the necessary condition for a functional to achieve a stationary value. This necessary condition on the functional is generally in the form of a differential equation with boundary conditions on the required function.

Consider the problem of finding a function $y(x)$ such that the function

$$I(y) = \int_a^b F(x, y, y')\, dx \ , \tag{4.12a}$$

subject to the boundary conditions

$$y(a) = A, \qquad y(b) = B \ , \tag{4.12b}$$

is rendered stationary. The integrand $F(x, y, y')$ is a given function of x, y, and $y' = dy/dx$. In Eq. (4.12a), $I(y)$ is called a *functional* or *variational* (or *stationary) principle*. The problem here is finding an extremizing function $y(x)$ for which the functional $I(y)$ has an extremum. Before attacking this problem, it is necessary that we introduce the operator δ, called the *variational symbol*.

The variation δy of a function $y(x)$ is an infinitesimal change in y for a fixed value of the independent variable x, i.e., for $\delta x = 0$. The variation δy of y vanishes at points where y is prescribed (since the prescribed value cannot be varied) and it is arbitrary elsewhere (see Fig. 4.1). Due to the change in y (i.e., $y \to y + \delta y$), there

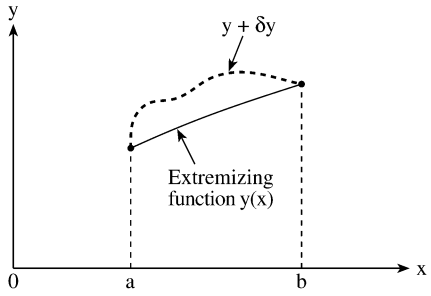

Figure 4.1
Variation of extremizing function with fixed ends.

is a corresponding change in F. The first variation of F at y is defined by

$$\delta F = \frac{\partial F}{\partial y}\delta y + \frac{\partial F}{\partial y'}\delta y' \tag{4.13}$$

This is analogous to the total differential of F,

$$dF = \frac{\partial F}{\partial x}\,dx + \frac{\partial F}{\partial y}\,dy + \frac{\partial F}{\partial y'}\,dy' \tag{4.14}$$

where $\delta x = 0$ since x does not change as y changes to $y + \delta y$. Thus, we note that the operator δ is similar to the differential operator. Therefore, if $F_1 = F_1(y)$ and $F_2 = F_2(y)$, then

$$(i) \qquad \delta\,(F_1 \pm F_2) = \delta F_1 \pm \delta F_2\,, \tag{4.15a}$$

$$(ii) \qquad \delta\,(F_1 F_2) = F_2 \delta F_1 + F_1 \delta F_2\,, \tag{4.15b}$$

$$(iii) \qquad \delta\left(\frac{F_1}{F_2}\right) = \frac{F_2 \delta F_1 - F_1 \delta F_2}{F_2^2}\,, \tag{4.15c}$$

$$(iv) \qquad \delta\,(F_1)^n = n\,(F_1)^{n-1}\,\delta F_1\,, \tag{4.15d}$$

$$(v) \qquad \frac{d}{dx}(\delta y) = \delta\left(\frac{dy}{dx}\right)\,, \tag{4.15e}$$

$$(vi) \qquad \delta \int_a^b y(x)\,dx = \int_a^b \delta y(x)\,dx \tag{4.15f}$$

A necessary condition for the function $I(y)$ in Eq. (4.12a) to have an extremum is that the variation vanishes, i.e.,

$$\boxed{\delta I = 0} \tag{4.16}$$

To apply this condition, we must be able to find the variation δI of I in Eq. (4.12a). To this end, let $h(x)$ be an increment in $y(x)$. For Eq. (4.12b) to be satisfied by $y(x) + h(x)$,

$$h(a) = h(b) = 0 \tag{4.17}$$

The corresponding increment in I in Eq. (4.12a) is

$$\begin{aligned}\Delta I &= I(y+h) - I(y) \\ &= \int_a^b \left[F\left(x, y+h, y'+h'\right) - F\left(x, y, y'\right)\right] dx\end{aligned}$$

On applying Taylor's expansion,

$$\begin{aligned}\Delta I &= \int_a^b \left[F_y\left(x, y, y'\right) h - F_{y'}\left(x, y, y'\right) h'\right] dx \\ &\quad + \text{higher order terms} \\ &= \delta I + O\left(h^2\right)\end{aligned}$$

where

$$\delta I = \int_a^b \left[F_y\left(x, y, y'\right) h - F_{y'}\left(x, y, y'\right) h'\right] dx$$

Integration by parts leads to

$$\delta I = \int_a^b \left[\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) \right] h\, dx + \left. \frac{\partial F}{\partial y'} h \right|_{x=0}^{x=b}$$

The last term vanishes since $h(b) = h(a) = 0$ according to Eq. (4.17). In order that $\delta I = 0$, the integrand must vanish, i.e.,

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$$

or

$$\boxed{F_y - \frac{d}{dx} F_{y'} = 0} \tag{4.18}$$

This is called *Euler's* (or *Euler-Lagrange) equation.* Thus a necessary condition for $I(y)$ to have an extremum for a given function $y(x)$ is that $y(x)$ satisfies Euler's equation.

This idea can be extended to more general cases. In the case considered so far, we have one dependent variable y and one independent variable x, i.e., $y = y(x)$. If we have one dependent variable u and two independent variables x and y, i.e., $u = u(x, y)$, then

$$I(u) = \int_S F\left(x, y, u, u_x, u_y\right)\, dS \tag{4.19}$$

where $u_x = \partial u/\partial x$, $u_y = \partial u/\partial y$, and $dS = dxdy$. The functional in Eq. (4.19) is stationary when $\delta I = 0$, and it is easily shown that the corresponding Euler's equation is [6]

$$\boxed{\frac{\partial F}{\partial u} - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u_y}\right) = 0} \tag{4.20}$$

Next we consider the case of two independent variables x and y and two dependent variables $u(x, y)$ and $v(x, y)$. The functional to be minimized is

$$I(u, v) = \int_S F\left(x, y, u, v, u_x, u_y, v_x, v_y\right)\, dS \tag{4.21}$$

The corresponding Euler's equation is

$$\boxed{\begin{aligned} &\frac{\partial F}{\partial u} - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u_y}\right) = 0 \qquad (4.22a)\\ &\frac{\partial F}{\partial v} - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial v_x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial v_y}\right) = 0 \qquad (4.22b) \end{aligned}}$$

Another case is when the functional depends on second- or higher-order derivatives. For example,

$$I(y) = \int_a^b F\left(x, y, y', y'', \ldots, y^{(n)}\right)\, dx \tag{4.23}$$

In this case, the corresponding Euler's equation is

$$\boxed{F_y - \frac{d}{dx}F_{y'} + \frac{d^2}{dx^2}F_{y''} - \frac{d^3}{dx^3}F_{y'''} + \cdots + (-1)^n \frac{d^n}{dx^n}F_{y^{(n)}} = 0} \tag{4.24}$$

Note that each of Euler's equations (4.18), (4.20), (4.22), and (4.24) is a differential equation.

Example 4.3

Given the functional

$$I(\Phi) = \int_S \left[\frac{1}{2}\left(\Phi_x^2 + \Phi_y^2\right) - f(x, y)\Phi\right] dxdy \,,$$

obtain the relevant Euler's equation. □

Solution

Let

$$F\left(x, y, \Phi, \Phi_x, \Phi_y\right) = \frac{1}{2}\left(\Phi_x^2 + \Phi_y^2\right) - f(x, y)\Phi$$

showing that we have two independent variables x and y and one dependent variable Φ. Hence, Euler's equation (4.20) becomes

$$-f(x, y) - \frac{\partial}{\partial x}\Phi_x - \frac{\partial}{\partial y}\Phi_y = 0$$

or

$$\Phi_{xx} + \Phi_{yy} = -f(x, y) \,,$$

i.e.,

$$\nabla^2\Phi = -f(x, y)$$

which is Poisson's equation. Thus, solving Poisson's equation is equivalent to finding Φ that extremizes the given functional $I(\Phi)$. ■

4.4 Construction of Functionals from PDEs

In the previous section, we noticed that Euler's equation produces the governing differential equation corresponding to a given functional or variational principle. Here we seek the inverse procedure of constructing a variational principle for a given differential equation. The procedure for finding the functional associated with the differential equation involves four basic steps [2, 7]:

- Multiply the operator equation $L\Phi = g$ (Euler's equation) with the variational $\delta\Phi$ of the dependent variable Φ and integrate over the domain of the problem.
- Use the divergence theorem or integration by parts to transfer the derivatives to variation $\delta\Phi$.
- Express the boundary integrals in terms of the specified boundary conditions.
- Bring the variational operator δ outside the integrals.

The procedure is best illustrated with an example. Suppose we are interested in finding the variational principle associated with the Poisson's equation

$$\nabla^2\Phi = -f(x, y) \tag{4.25}$$

which is the converse of what we did in Example 4.3. After taking step 1, we have

$$\begin{aligned}\delta I &= \iint \left[-\nabla^2\Phi - f\right]\delta\Phi\, dxdy = 0\\ &= -\iint \nabla^2\Phi\delta\Phi\, dxdy - \iint f\delta\Phi\, dxdy\end{aligned}$$

This can be evaluated by applying divergence theorem or integrating by parts. To integrate by parts, let $u = \delta\Phi, dv = \dfrac{\partial}{\partial x}\left(\dfrac{\partial\Phi}{\partial x}\right)dx$ so that $du = \dfrac{\partial}{\partial x}\delta\Phi\, dx, v = \dfrac{\partial\Phi}{\partial x}$ and

$$-\int\left[\int\frac{\partial}{\partial x}\left(\frac{\partial\Phi}{\partial x}\right)\delta\Phi\, dx\right]dy = -\int\left[\delta\Phi\frac{\partial\Phi}{\partial x} - \int\frac{\partial\Phi}{\partial x}\frac{\partial}{\partial x}\delta\Phi\, dx\right]dy$$

Thus

$$\begin{aligned}\delta I &= \iint\left[\frac{\partial\Phi}{\partial x}\frac{\partial}{\partial x}\delta\Phi + \frac{\partial\Phi}{\partial y}\frac{\partial}{\partial y}\delta\Phi - \delta f\Phi\right]dxdy\\ &\quad -\int\delta\Phi\frac{\partial\Phi}{\partial x}\,dy - \int\delta\Phi\frac{\partial\Phi}{\partial y}\,dx\\ \delta I &= \frac{\delta}{2}\iint\left[\left(\frac{\partial\Phi}{\partial x}\right)^2 + \left(\frac{\partial\Phi}{\partial y}\right)^2 - 2f\Phi\right]dxdy\\ &\quad -\delta\int\Phi\frac{\partial\Phi}{\partial x}\,dy - \delta\int\Phi\frac{\partial\Phi}{\partial y}\,dx\end{aligned} \tag{4.26}$$

The last two terms vanish if we assume either the homogeneous Dirichlet or Neumann conditions at the boundaries. Hence

$$\delta I = \delta\iint\frac{1}{2}\left[\Phi_x^2 + \Phi_y^2 - 2\Phi f\right]dxdy\ ,$$

i.e.,

$$I(\Phi) = \frac{1}{2}\iint \left[\Phi_x^2 + \Phi_y^2 - 2\Phi f\right] dxdy \tag{4.27}$$

as expected.

Rather than following the four steps listed above to find the function $I(\Phi)$ corresponding to the operator equation (4.8), an alternative approach is provided by Mikhlin [1, pp. 74–78]. According to Mikhlin, if L in Eq. (4.8) is real, self-adjoint, and positive definite, the solution of Eq. (4.8) minimizes the functional

$$\boxed{I(\Phi) = \langle L\Phi, \Phi\rangle - 2\langle \Phi, g\rangle} \tag{4.28}$$

(See Prob. 4.6 for a proof.) Thus Eq. (4.27), for example, can be obtained from Eq. (4.25) by applying Eq. (4.28). This approach has been applied to derive variational solutions of integral equations [8].

Other systematic approaches for the derivation of variational principles for EM problems include Hamilton's principle or the principle of least action [9, 10], Lagrange multipliers [10]–[14], and a technique described as variational electromagnetics [15, 16]. The method of Lagrange undetermined multipliers is particularly useful for deriving a functional for a PDE whose arguments are constrained. Table 4.1 provides the variational principles for some differential equations commonly found in EM-related problems.

Table 4.1 Variational Principle Associated with Common PDEs in EM[1]

Name of equation	Partial Differential Equation (PDE)	Variational principle
Inhomogeneous wave equation	$\nabla^2\Phi + k^2\Phi = g$	$I(\Phi) = \frac{1}{2}\int_v \left[\lvert\nabla\Phi\rvert^2 - k^2\Phi^2 + 2g\Phi\right] dv$
Homogeneous wave equation	$\nabla^2\Phi + k^2\Phi = 0$	$I(\Phi) = \frac{1}{2}\int_v \left[\lvert\nabla\Phi\rvert^2 - k^2\Phi^2\right] dv$
	or	
	$\nabla^2\Phi - \frac{1}{u^2}\Phi_{tt} = 0$	$I(\Phi) = \frac{1}{2}\int^{t_o}\int_v \left[\lvert\nabla\Phi\rvert^2 - \frac{1}{u^2}\Phi_t^2\right] dv\,dt$
Diffusion equation	$\nabla^2\Phi - k\Phi_t = 0$	$I(\Phi) = \frac{1}{2}\int^{t_o}\int_v \left[\lvert\nabla\Phi\rvert^2 - k\Phi\Phi_t\right] dv\,dt$
Poisson's equation	$\nabla^2\Phi = g$	$I(\Phi) = \frac{1}{2}\int_v \left[\lvert\nabla\Phi\rvert^2 + 2g\Phi\right] dv$
Laplace's equation	$\nabla^2\Phi = 0$	$I(\Phi) = \frac{1}{2}\int_v \left[\lvert\nabla\Phi\rvert^2\right] dv$

[1] Note that $|\nabla\Phi|^2 = \nabla\Phi\cdot\nabla\Phi = \Phi_x^2 + \Phi_y^2 + \Phi_z^2$.

Example 4.4
Find the functional for the ordinary differential equation

$$y'' + y + x = 0, \qquad 0 < x < 1$$

subject to $y(0) = y(1) = 0$. □

Solution
Given that

$$\frac{d^2y}{dx^2} + y + x = 0, \qquad 0 < x < 1\,,$$

we obtain

$$\begin{aligned}\delta I &= \int_0^1 \left(\frac{d^2y}{dx^2} + y + x\right)\delta y\, dx = 0\\ &= \int_0^1 \frac{d^2y}{dx^2}\delta y\, dx + \int_0^1 y\,\delta y\, dx + \int_0^1 x\,\delta y\, dx\end{aligned}$$

Integrating the first term by parts,

$$\delta I = \delta y\, \frac{dy}{dx}\bigg|_{x=0}^{x=1} - \int_0^1 \frac{dy}{dx}\frac{d}{dx}\delta y + \int_0^1 \frac{1}{2}\delta\left(y^2\right)\, dx + \delta\int_0^1 xy\, dx$$

Since y is fixed at $x = 0, 1$, $\delta y(1) = \delta y(0) = 0$. Hence

$$\begin{aligned}\delta I &= -\delta\int_0^1 \frac{1}{2}\left(\frac{dy}{dx}\right)^2 dx + \frac{1}{2}\delta\int_0^1 y^2\, dx + \delta\int_0^1 xy\, dx\\ &= \frac{\delta}{2}\int_0^1 \left[-y'^2 + y^2 + 2xy\right] dx\end{aligned}$$

or

$$I(y) = \frac{1}{2}\int_0^1 \left[-y'^2 + y^2 + 2xy\right] dx$$

Check: Taking $F(x, y, y') = y'^2 - y^2 - 2xy$, Euler's equation $F_y - \dfrac{d}{dx}F_{y'} = 0$ gives the differential equation

$$y'' + y + x = 0 \qquad \blacksquare$$

4.5 Rayleigh-Ritz Method

The Rayleigh-Ritz method is the direct variational method for minimizing a given functional. It is direct in that it yields a solution to the variational problem without

recourse to the associated differential equation [17]. In other words, it is the direct application of variational principles discussed in the previous sections. The method was first presented by Rayleigh in 1877 and extended by Ritz in 1909. Without loss of generality, let the associated variational principle be

$$I(\Phi) = \int_S F\left(x, y, \Phi, \Phi_x, \Phi_y\right) dS \tag{4.29}$$

Our objective is to minimize this integral. In the Rayleigh-Ritz method, we select a linearly independent set of functions called *expansion functions* (or *basis functions)* u_n and construct an approximate solution to Eq. (4.29), satisfying some prescribed boundary conditions. The solution is in the form of a finite series

$$\widetilde{\Phi} \simeq \sum_{n=1}^{N} a_n u_n + u_o \tag{4.30}$$

where u_o meets the nonhomogeneous boundary conditions, and u_n satisfies homogeneous boundary conditions. a_n are expansion coefficients to be determined and $\widetilde{\Phi}$ is an approximate solution to Φ (the exact solution). We substitute Eq. (4.30) into Eq. (4.29) and convert the integral $I(\Phi)$ into a function of N coefficients $a_1, a_2, \ldots, a_N$, i.e.,

$$I(\Phi) = I\left(a_1, a_2, \ldots, a_N\right)$$

The minimum of this function is obtained when its partial derivatives with respect to each coefficient is zero:

$$\frac{\partial I}{\partial a_1} = 0, \quad \frac{\partial I}{\partial a_2} = 0, \ldots, \frac{\partial I}{\partial a_N} = 0$$

or

$$\boxed{\frac{\partial I}{\partial a_n} = 0, \quad n = 1, 2, \ldots, N} \tag{4.31}$$

Thus we obtain a set of N simultaneous equations. The system of linear algebraic equations obtained is solved to get a_n, which are finally substituted into the approximate solution of Eq. (4.30). In the approximate solution of Eq. (4.30), if $\widetilde{\Phi} \to \Phi$ as $N \to \infty$ in some sense, then the procedure is said to *converge* to the exact solution.

An alternative, perhaps easier, procedure to determine the expansion coefficients a_n is by solving a system of simultaneous equations obtained as follows [4, 18]. Substituting Eq. (4.30) (ignoring u_o since it can be lumped with the right-hand side of the equation) into Eq. (4.28) yields

$$\begin{aligned} I &= \left\langle \sum_{m=1}^{N} a_m L u_m, \sum_{n=1}^{N} a_n u_n \right\rangle - 2\left\langle \sum_{m=1}^{N} a_m u_m, g \right\rangle \\ &= \sum_{m=1}^{N}\sum_{n=1}^{N} \langle L u_m, u_n \rangle a_n a_m - 2\sum_{m=1}^{N} \langle u_m, g \rangle a_m \end{aligned}$$

Expanding this into powers of a_m results in

$$I = \langle Lu_m, u_m\rangle a_m^2 + \sum_{n\neq m}^{N} \langle Lu_m, u_n\rangle a_m a_n + \sum_{k\neq m}^{N} \langle Lu_k, u_m\rangle a_k a_m - 2\langle g, u_m\rangle a_m + \text{terms not containing } a_m \tag{4.32}$$

Assuming L is self-adjoint and replacing k with n in the second summation,

$$I = \langle Lu_m, u_m\rangle a_m^2 + 2\sum_{n\neq m}^{N} \langle Lu_m, u_n\rangle a_n a_m - 2\langle g, u_m\rangle a_m + \cdots \tag{4.33}$$

Since we are interested in selecting a_m such that I is minimized, Eq. (4.33) must satisfy Eq. (4.31). Thus differentiating Eq. (4.33) with respect to a_m and setting the result equal to zero leads to

$$\sum_{n=1}^{N} \langle Lu_m, u_n\rangle a_n = \langle g, u_m\rangle, \qquad m = 1, 2, \ldots, N \tag{4.34}$$

which can be put in matrix form as

$$\boxed{\begin{bmatrix} \langle Lu_1, u_1\rangle & \langle Lu_1, u_2\rangle & \cdots & \langle Lu_1, u_N\rangle \\ \vdots & & & \vdots \\ \langle Lu_N, u_1\rangle & \langle Lu_N, u_2\rangle & \cdots & \langle Lu_N, u_N\rangle \end{bmatrix} \begin{bmatrix} a_1 \\ \vdots \\ a_N \end{bmatrix} = \begin{bmatrix} \langle g, u_1\rangle \\ \vdots \\ \langle g, u_N\rangle \end{bmatrix}} \tag{4.35a}$$

or

$$[A][X] = [B] \tag{4.35b}$$

where $A_{mn} = \langle Lu_m, u_n\rangle$, $B_m = \langle g, u_m\rangle$, $X_n = a_n$. Solving for $[X]$ in Eq. (4.35) and substituting a_m in Eq. (4.30) gives the approximate solution $\widetilde{\Phi}$. Equation (4.35) is called the *Rayleigh-Ritz system.*

We are yet to know how the expansion functions are selected. They are selected to satisfy the prescribed boundary conditions of the problem. u_o is chosen to satisfy the inhomogeneous boundary conditions, while $u_n (n = 1, 2, \ldots, N)$ are selected to satisfy the homogeneous boundary conditions. If the prescribed boundary conditions are all homogeneous (Dirichlet conditions), $u_o = 0$. The next section will discuss more on the selection of the expansion functions.

The Rayleigh-Ritz method has two major limitations. First, the variational principle in Eq. (4.29) may not exist in some problems such as in nonself-adjoint equations (odd order derivatives). Second, it is difficult, if not impossible, to find the functions u_o satisfying the global boundary conditions for the domains with complicated geometries [19].

Example 4.5

Use the Rayleigh-Ritz method to solve the ordinary differential equation:

$$\Phi'' + 4\Phi - x^2 = 0, \qquad 0 < x < 1$$

subject to $\Phi(0) = 0 = \Phi(1)$. □

Solution

The exact solution is

$$\Phi(x) = \frac{\sin 2(1-x) - \sin 2x}{8 \sin 2} + \frac{x^2}{4} - \frac{1}{8}$$

The variational principle associated with $\Phi'' + 4\Phi - x^2 = 0$ is

$$I(\Phi) = \int_0^1 \left[(\Phi')^2 - 4\Phi^2 + 2x^2\Phi \right] dx \tag{4.36}$$

This is readily verified using Euler's equation. We let the approximate solution be

$$\widetilde{\Phi} = u_o + \sum_{n=1}^{N} a_n u_n \tag{4.37}$$

where $u_o = 0, u_n = x^n(1-x)$ since $\Phi(0) = 0 = \Phi(1)$ must be satisfied. (This choice of u_n is not unique. Other possible choices are $u_n = x(1-x^n)$ and $u_n = \sin n\pi x$. Note that each choice satisfies the prescribed boundary conditions.) Let us try different values of N, the number of expansion coefficients. We can find the expansion coefficients a_n in two ways: using the functional directly as in Eq. (4.31) or using the Rayleigh-Ritz system of Eq. (4.35).

Method 1

For $N = 1$, $\widetilde{\Phi} = a_1 u_1 = a_1 x(1-x)$. Substituting this into Eq. (4.36) gives

$$\begin{aligned} I(a_1) &= \int_0^1 \left[a_1^2(1-2x)^2 - 4a_1^2\left(x - x^2\right)^2 + 2a_1 x^3(1-x) \right] dx \\ &= \frac{1}{5}a_1^2 + \frac{1}{10}a_1 \end{aligned}$$

$I(a_1)$ is minimum when

$$\frac{\partial I}{\partial a_1} = 0 \quad \rightarrow \quad \frac{2}{5}a_1 + \frac{1}{10} = 0 \quad \text{or} \quad a_1 = -\frac{1}{4}$$

Hence the quadratic approximate solution is

$$\widetilde{\Phi} = -\frac{1}{4}x(1-x) \tag{4.38}$$

For $N = 2$, $\widetilde{\Phi} = a_1 u_1 + a_2 u_2 = a_1 x(1-x) + a_2 x^2(1-x)$. Substituting $\widetilde{\Phi}$ into Eq. (4.36),

$$
\begin{aligned}
I(a_1, a_2) &= \int_0^1 \left[\left[a_1(1-2x) + a_2\left(2x - 3x^2\right) \right]^2 - 4\left[a_1\left(x - x^2\right) + a_2\left(x^2 - x^3\right) \right]^2 \right. \\
&\quad \left. + 2a_1 x^2\left(x - x^2\right) + 2a_2 x^2\left(x^2 - x^3\right) \right] dx \\
&= \frac{1}{5}a_1^2 + \frac{2}{21}a_2^2 + \frac{1}{5}a_1 a_2 + \frac{1}{10}a_1 + \frac{1}{15}a_2 \\
&\qquad \frac{\partial I}{\partial a_1} = 0 \quad \rightarrow \quad \frac{2}{5}a_1 + \frac{1}{5}a_2 + \frac{1}{10} = 0
\end{aligned}
$$

or

$$
4a_1 + 2a_2 = -1 \tag{4.39a}
$$

$$
\frac{\partial I}{\partial a_2} = 0 \quad \rightarrow \quad \frac{4}{21}a_1 + \frac{1}{5}a_2 + \frac{1}{15} = 0
$$

or

$$
21a_1 + 20a_2 = -7 \tag{4.39b}
$$

Solving Eq. (4.39) gives

$$
a_1 = -\frac{6}{38}, \qquad a_2 = -\frac{7}{38}
$$

and hence the cubic approximate solution is

$$
\widetilde{\Phi} = -\frac{6}{38}x(1-x) - \frac{7}{38}x^2(1-x)
$$

or

$$
\widetilde{\Phi} = \frac{x}{38}\left(7x^2 - x - 6\right)
$$

Method 2

We now determine a_m using Eq. (4.35). From the given differential equation,

$$
L = \frac{d^2}{dx^2} + 4, \qquad g = x^2
$$

Hence

$$
\begin{aligned}
A_{mn} &= \langle Lu_m, u_n \rangle = \langle u_m, Lu_n \rangle \\
&= \int_0^1 x^m(1-x)\left[\left(\frac{d^2}{dx^2} + 4 \right) x^n(1-x) \right] dx \,, \\
A_{mn} &= \frac{n(n-1)}{m+n-1} - \frac{2n^2}{m+n} + \frac{n(n+1)+4}{m+n+1} - \frac{8}{m+n+2} + \frac{4}{m+n+3} \,, \\
B_n &= \langle g, u_n \rangle = \int_0^1 x^2 n^n (1-x)\, dx = \frac{1}{n+3} - \frac{1}{n+4}
\end{aligned}
$$

When $N = 1$, $A_{11} = -\frac{1}{5}$, $B_1 = \frac{1}{20}$, i.e.,

$$-\frac{1}{5}a_1 = \frac{1}{20} \quad \rightarrow \quad a_1 = -\frac{1}{4}$$

as before. When $N = 2$,

$$A_{11} = -\frac{1}{5},\ A_{12} = A_{21} = -\frac{1}{10},\ A_{22} = -\frac{2}{21},\ B_1 = \frac{1}{20},\ B_2 = \frac{1}{30}$$

Hence

$$\begin{bmatrix} -\frac{1}{5} & -\frac{1}{10} \\ -\frac{1}{10} & -\frac{2}{21} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{20} \\ \frac{1}{30} \end{bmatrix}$$

which gives $a_1 = -\frac{6}{38}, a_2 = -\frac{7}{38}$ as obtained previously. When $N = 3$,

$$A_{13} = A_{31} = -\frac{13}{210},\ A_{23} = A_{32} = -\frac{28}{105},\ A_{33} = -\frac{22}{315},\ B_3 = \frac{1}{42},$$

i.e.,

$$\begin{bmatrix} -\frac{1}{5} & -\frac{1}{10} & -\frac{13}{210} \\ -\frac{1}{10} & -\frac{2}{21} & -\frac{28}{105} \\ -\frac{13}{210} & -\frac{28}{105} & -\frac{22}{315} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{20} \\ \frac{1}{30} \\ \frac{1}{42} \end{bmatrix}$$

from which we obtain

$$a_1 = -\frac{6}{38}, \qquad a_2 = -\frac{7}{38}, \qquad a_3 = 0$$

showing that we obtain the same solution as for the case $N = 2$. For different values of $x, 0 < x < 1$, the Rayleigh-Ritz solution is compared with the exact solution in Table 4.2. ∎

Table 4.2 Comparison of Exact Solution with the Rayleigh-Ritz Solution of $\Phi'' + 4\Phi - x^2 = 0, \Phi(0) = 0 = \Phi(1)$

x	Exact solution	Rayleigh-Ritz $N = 1$	Solution $N = 2$
0.0	0.0	0.0	0.0
0.2	−0.0301	−0.0400	−0.0312
0.4	−0.0555	−0.0600	−0.0556
0.6	−0.0625	−0.0625	−0.0644
0.8	−0.0489	−0.0400	−0.0488
1.0	0.0	0.0	0.0

Example 4.6

Using the Rayleigh-Ritz method, solve Poisson's equation:

$$\nabla^2 V = -\rho_o, \qquad \rho_o = \text{constant}$$

in a square $-1 \le x, y \le 1$, subject to the homogeneous boundary conditions $V(x, \pm 1) = 0 = V(\pm 1, y)$. □

Solution

Due to the symmetry of the problem, we choose the basis functions as

$$u_{mn} = \left(1 - x^2\right)\left(1 - y^2\right)\left(x^{2m} y^{2n} + x^{2n} y^{2m}\right), \quad m, n = 0, 1, 2, \ldots$$

Hence

$$\widetilde{\Phi} = \left(1 - x^2\right)\left(1 - y^2\right)\left[a_1 + a_2\left(x^2 + y^2\right) + a_3 x^2 y^2 + a_4\left(x^4 + y^4\right) + \cdots\right]$$

Case 1: When $m = n = 0$, we obtain the first approximation ($N = 1$) as

$$\widetilde{\Phi} = a_1 u_1$$

where $u_1 = (1 - x^2)(1 - y^2)$.

$$\begin{aligned} A_{11} &= \langle L u_1, u_1 \rangle = \int_{-1}^{1}\int_{-1}^{1}\left(\frac{\partial^2 u_1}{\partial x^2} + \frac{\partial^2 u_1}{\partial y^2}\right) u_1 \, dxdy \\ &= -8\int_0^1\int_0^1 \left(2 - x^2 - y^2\right)\left(1 - x^2\right)\left(1 - y^2\right) dxdy \\ &= -\frac{256}{45}, \\ B_1 &= \langle g, u_1 \rangle = -\int_{-1}^{1}\int_{-1}^{1}\left(1 - x^2\right)\left(1 - y^2\right)\rho_o \, dxdy = -\frac{16\rho_o}{9} \end{aligned}$$

Hence

$$-\frac{256}{45}a_1 = -\frac{16}{9}\rho_o \quad \rightarrow \quad a_1 = \frac{5}{16}\rho_o$$

and

$$\widetilde{\Phi} = \frac{5}{16}\rho_o\left(1 - x^2\right)\left(1 - y^2\right)$$

Case 2: When $m = n = 1$, we obtain the second order approximation ($N = 2$) as

$$\widetilde{\Phi} = a_1 u_1 + a_2 u_2$$

where $u_1 = (1-x^2)(1-y^2)$, $u_2 = (1-x^2)(1-y^2)(x^2+y^2)$. A_{11} and B_1 are the same as in case 1.

$$A_{12} = A_{21} = \langle Lu_1, u_2 \rangle = -\frac{1024}{525},$$
$$A_{22} = \langle Lu_2, u_2 \rangle = -\frac{11264}{4725},$$
$$B_2 = \langle g, u_2 \rangle = -\frac{32}{45}\rho_o$$

Hence

$$\begin{bmatrix} -\frac{256}{45} & -\frac{1024}{525} \\ -\frac{1024}{525} & -\frac{11264}{4725} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} -\frac{16}{9}\rho_o \\ -\frac{32}{45}\rho_o \end{bmatrix}$$

Solving this yields

$$a_1 = \frac{1295}{4432}\rho_o = 0.2922\rho_o, \quad a_2 = \frac{525}{8864}\rho_o = 0.0592\rho_o$$

and

$$\widetilde{\Phi} = \left(1-x^2\right)\left(1-y^2\right)\left(0.2922 + 0.0592\left(x^2+y^2\right)\right)\rho_o \quad \blacksquare$$

4.6 Weighted Residual Method

As noted earlier, the Rayleigh-Ritz method is applicable when a suitable functional exists. In cases where such a functional cannot be found, we apply one of the techniques collectively referred to as the *method of weighted residuals*. The method is more general and has wider application than the Rayleigh-Ritz method because it is not limited to a class of variational problems.

Consider the operator equation

$$L\Phi = g \tag{4.40}$$

In the weighted residual method, the solution to Eq. (4.40) is approximated, in the same manner as in the Rayleigh-Ritz method, using the expansion functions, u_n, i.e.,

$$\widetilde{\Phi} = \sum_{n=1}^{N} a_n u_n \tag{4.41a}$$

where a_n are the expansion coefficients. We seek to make

$$L\widetilde{\Phi} \approx g \tag{4.41b}$$

Substitution of the approximate solution in the operator equation results in a *residual* R (an error in the equation), i.e.,

$$\boxed{R = L(\widetilde{\Phi} - \Phi) = L\widetilde{\Phi} - g} \tag{4.42}$$

In the weighted residual method, the weighting functions w_m (which, in general, are not the same as the expansion functions u_n) are chosen such that the integral of a weighted residual of the approximation is zero, i.e.,

$$\int w_m R \, dv = 0$$

or

$$\boxed{\langle w_m, R \rangle = 0} \tag{4.43}$$

If a set of *weighting functions* $\{w_m\}$ (also known as *testing functions*) are chosen and the inner product of Eq. (4.41) is taken for each w_m, we obtain

$$\sum_{n=1}^{N} a_n \langle w_m, Lu_n \rangle = \langle w_m, g \rangle, \qquad m = 1, 2, \ldots, N \tag{4.44}$$

The system of linear equations (4.42) can be cast into matrix form as

$$[A][X] = [B] \tag{4.45}$$

where $A_{mn} = \langle w_m, Lu_n \rangle$, $B_m = \langle w_m, g \rangle$, $X_n = a_n$. Solving for $[X]$ in Eq. (4.45) and substituting for a_n in Eq. (4.41a) gives the approximate solution to Eq. (4.40). However, there are different ways of choosing the weighting functions w_m leading to:

- collocation (or point-matching method),
- subdomain method,
- Galerkin method,
- least squares method.

4.6.1 Collocation Method

We select the Dirac delta function as the weighting function, i.e.,

$$w_m(\mathbf{r}) = \delta(\mathbf{r} - \mathbf{r}_m) = \begin{cases} 1, & \mathbf{r} = \mathbf{r}_m \\ 0, & \mathbf{r} \neq \mathbf{r}_m \end{cases} \tag{4.46}$$

Substituting Eq. (4.46) into Eq. (4.43) results in

$$R(\mathbf{r}) = 0 \tag{4.47}$$

Thus we select as many collocation (or matching) points in the interval as there are unknown coefficients a_n and make the residual zero at those points. This is equivalent to enforcing

$$\sum_{n=1}^{N} La_n u_n = g \tag{4.48}$$

at discrete points in the region of interest, generally where boundary conditions must be met. Although the point-matching method is the simplest specialization for computation, it is not possible to determine in advance for a particular operator equation what collocation points would be suitable. An accurate result is ensured only if judicious choice of the match points is taken. (This will be illustrated in Example 4.7.) It is important to note that the finite difference method is a particular case of collocation with locally defined expansion functions [20]. The validity and legitimacy of the point-matching technique are discussed in [21, 22].

4.6.2 Subdomain Method

We select weighting functions w_m, each of which exists only over subdomains of the domain of Φ. Typical examples of such functions for one-dimensional problems are illustrated in Fig. 4.2 and defined as follows.

(1) piecewise uniform (or pulse) function:

$$w_m(x) = \begin{cases} 1, & x_{m-1} < x < x_{m+1} \\ 0, & \text{otherwise} \end{cases} \tag{4.49a}$$

(2) piecewise linear (or triangular) function:

$$w_m(x) = \begin{cases} \dfrac{\Delta - |x - x_m|}{\Delta}, & x_{m-1} < x < x_{m+1} \\ 0, & \text{otherwise} \end{cases} \tag{4.49b}$$

(3) piecewise sinusoidal function:

$$w_m(x) = \begin{cases} \dfrac{\sin k(x - |x - x_m|)}{\Delta}, & x_{m-1} < x < x_{m+1} \\ 0, & \text{otherwise} \end{cases} \tag{4.49c}$$

Using the unit pulse function, for example, is equivalent to dividing the domain of Φ into as many subdomains as there are unknown terms and letting the average value of R over such subdomains vanish.

4.6.3 Galerkin Method

We select basis functions as the weighting function, i.e., $w_m = u_m$. When the operator is a linear differential operator of even order, the Galerkin method[1] reduces

[1] The Galerkin method was developed by the Russian engineer B.G. Galerkin in 1915.

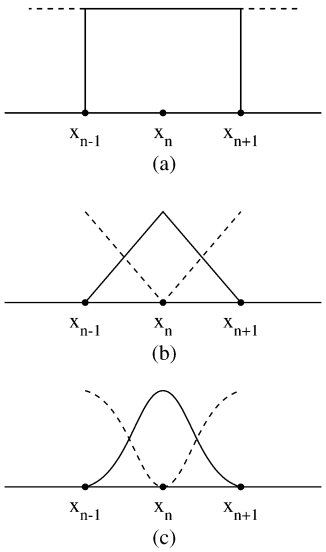

Figure 4.2

Typical subdomain weighting functions: (a) piecewise uniform function, (b) piecewise linear function, (c) piecewise sinusoidal function.

to the Rayleigh-Ritz method. This is due to the fact that the differentiation can be transferred to the weighting functions and the resulting coefficient matrix $[A]$ will be symmetric [7]. In order for the Galerkin method to be applicable, the operator must be of a certain type. Also, the expansion function u_n must span both the domain and the range of the operator.

4.6.4 Least Squares Method

This involves minimizing the integral of the square of the residual, i.e.,

$$\frac{\partial}{\partial a_m} \int R^2 \, dv = 0$$

or

$$\int \frac{\partial R}{\partial a_m} R \, dv = 0 \tag{4.50}$$

Comparing Eq. (4.50) with Eq. (4.43) shows that we must choose

$$w_m = \frac{\partial R}{\partial a_m} = L u_m \tag{4.51}$$

This may be viewed as requiring that

$$\frac{1}{2} \int R^2 \, dv$$

be minimum. In other words, the choice of w_m corresponds to minimizing the mean square residual. It should be noted that the least squares method involves higher order derivatives which will, in general, lead to a better convergence than the Rayleigh-Ritz method or Galerkin method, but it has the disadvantage of requiring higher order weighting functions [19].

The concept of convergence discussed in the previous section applies here as well. That is, if the approximate solution $\widetilde{\Phi}$ were to converge to the exact solution Φ as $N \to \infty$, the residual must approach zero as $N \to \infty$. Otherwise, the sequence of approximate solutions may not converge to any meaningful result.

The inner product involved in applying a weighted residual method can sometimes be evaluated analytically, but in most practical situations it is evaluated numerically. Due to a careless evaluation of the inner product, one may think that the least squares technique is being used when the resulting solution is identical to a point-matching solution. To avoid such erroneous results or conclusions, one must be certain that the overall number of points involved in the numerical integration is not smaller than the number of unknowns, N, involved in the weighted residual method [23].

The accuracy and efficiency of a weighted residual method is largely dependent on the selection of expansion functions. The solution may be exact or approximate depending on how we select the expansion and weighting functions [17]. The criteria for selecting expansion and weighting functions in a weighted residual method are provided in the work of Sarkar and others [24]–[27]. We summarize their results here. The expansion functions u_n are selected to satisfy the following requirements [27]:

(1) The expansion functions should be in the domain of the operator L in some sense, i.e., they should satisfy the differentiability criterion and they must satisfy the boundary conditions for the operator. It is not necessary for each expansion function to satisfy exactly the boundary conditions. What is required is that the total solution must satisfy the boundary conditions at least in some distributional sense. The same holds for the differentiability conditions.

(2) The expansion functions must be such that $L\,u_n$ form a complete set for the range of the operator. It really does not matter whether the expansion functions are complete in the domain of the operator. What is important is that u_n must be chosen in such a way that $L\,u_n$ is complete. This will be illustrated in Example 4.8.

From a mathematical point of view, the choice of expansion functions does not depend on the choice of weighting functions. It is required that the weighting functions w_n must take the difference $\Phi - \widetilde{\Phi}$ small. For the Galerkin method to be applicable, the expansion functions u_n must span both the domain and the range of the operator. For the least squares method, the weighting functions are already presented and defined by $L\,u_n$. It is necessary that $L\,u_n$ form a complete set. The least squares technique mathematically and numerically is one of the safest techniques to utilize when very little is known about the nature of the operator and the exact solution.

Example 4.7

Find an approximate solution to

$$\Phi'' + 4\Phi - x^2 = 0, \qquad 0 < x < 1,$$

with $\Phi(0) = 0$, $\Phi'(1) = 1$, using the method of weighted residuals. □

Solution

The exact solution is

$$\Phi(x) = \frac{\cos 2(x-1) + 2\sin 2x}{8\cos 2} - \frac{x^2}{4} - \frac{1}{8} \tag{4.52}$$

Let the approximate solution be

$$\widetilde{\Phi} = u_0 + \sum_{n=1}^{N} a_n u_n \tag{4.53}$$

The boundary conditions can be decomposed into two parts:

(1) homogeneous part $\rightarrow$ $\Phi(0) = 0$, $\Phi'(0) = 0$,

(2) inhomogeneous part $\rightarrow$ $\Phi'(1) = 1$.

We choose u_0 to satisfy the inhomogeneous boundary condition. A reasonable choice is

$$u_0 = x \tag{4.54a}$$

We choose $u_n (n = 1, 2, \ldots, N)$ to satisfy the homogeneous boundary condition. Suppose we select

$$u_n(x) = x^n \left(x - \frac{n+1}{n} \right) \tag{4.54b}$$

Thus, if we take $N = 2$, the approximate solution is

$$\begin{aligned} \widetilde{\Phi} &= u_0 + a_1 u_1 + a_2 u_2 \\ &= x + a_1 x(x-2) + a_2 x^2 (x - 3/2) \end{aligned} \tag{4.55}$$

where the expansion coefficients, a_1 and a_2, are to be determined. We find the residual R using Eq. (4.42), namely,

$$\begin{aligned} R &= L\widetilde{\Phi} - g \\ &= \left(\frac{d^2}{dx^2} + 4 \right) \widetilde{\Phi} - x^2 \\ &= a_1 \left(4x^2 - 8x + 2 \right) + a_2 \left(4x^3 - 6x^2 + 6x - 3 \right) - x^2 + 4x \end{aligned} \tag{4.56}$$

We now apply each of the four techniques discussed and compare the solutions.

Method 1: (collocation or point-matching method)
Since we have two unknowns a_1 and a_2, we select two match points, at $x = \frac{1}{3}$ and $x = \frac{2}{3}$, and set the residual equal to zero at those points, i.e.,

$$R\left(\frac{1}{3}\right) = 0 \quad \rightarrow \quad 6a_1 + 41a_2 = 33$$
$$R\left(\frac{2}{3}\right) = 0 \quad \rightarrow \quad 42a_1 + 13a_2 = 60$$

Solving these equations,

$$a_1 = \frac{677}{548}, \qquad a_2 = \frac{342}{548}$$

Substituting a_1 and a_2 into Eq. (4.55) gives

$$\widetilde{\Phi}(x) = -1.471x + 0.2993x^2 + 0.6241x^3 \tag{4.57}$$

To illustrate the dependence of the solution on the match points, suppose we select $x = \frac{1}{4}$ and $x = \frac{3}{4}$ as the match points. Then

$$R\left(\frac{1}{4}\right) = 0 \quad \rightarrow \quad -4a_1 + 29a_2 = 15$$
$$R\left(\frac{3}{4}\right) = 0 \quad \rightarrow \quad 28a_1 + 3a_2 = 39$$

Solving for a_1 and a_2, we obtain

$$a_1 = \frac{543}{412}, \qquad a_2 = \frac{288}{412}$$

with the approximate solution

$$\widetilde{\Phi}(x) = -1.636x + 0.2694x^2 + 0.699x^3 \tag{4.58}$$

We will refer to the solutions in Eqs. (4.57) and (4.58) as collocation 1 and collocation 2, respectively. It is evident from Table 4.3 that collocation 2 is more accurate than collocation 1.

Method 2: (subdomain method)
Divide the interval $0 < x < 1$ into two segments since we have two unknowns a_1 and a_2. We select pulse functions as weighting functions:

$$w_1 = 1, \qquad 0 < x < \frac{1}{2},$$
$$w_2 = 1, \qquad \frac{1}{2} < x < 1$$

so that

$$\int_0^{1/2} w_1 R\,dx = 0 \quad \rightarrow \quad -8a_1 + 45a_2 = 22$$
$$\int_{1/2}^{1} w_2 R\,dx = 0 \quad \rightarrow \quad 40a_1 + 3a_2 = 58$$

Solving the two equations gives

$$a_1 = \frac{53}{38}, \qquad a_2 = \frac{28}{38}$$

and hence Eq. (4.55) becomes

$$\widetilde{\Phi}(x) = -1.789x + 0.2895x^2 + 0.7368x^3 \tag{4.59}$$

Method 3: (Galerkin method)
In this case, we select $w_m = u_m$, i.e.,

$$w_1 = x(x-2), \qquad w_2 = x^2(x - 3/2)$$

We now apply Eq. (4.43), namely, $\int w_m R\,dx = 0$. We obtain

$$\int_0^1 \left(x^2 - 2x\right) R\,dx = 0 \quad \rightarrow \quad 24a_1 + 11a_2 = 41$$
$$\int_0^1 \left(x^3 - \frac{3}{2}x^2\right) R\,dx = 0 \quad \rightarrow \quad 77a_1 + 15a_2 = 119$$

Solving these leads to

$$a_1 = \frac{694}{487}, \qquad a_2 = \frac{301}{487}$$

Substituting a_1 and a_2 into Eq. (4.55) gives

$$\widetilde{\Phi}(x) = -1.85x + 0.4979x^2 + 0.6181x^3 \tag{4.60}$$

Method 4: (least squares method)
For this method, we select $w_m = \dfrac{\partial R}{\partial a_m}$, i.e.,

$$w_1 = 4x^2 - 8x + 2, \qquad w_2 = 4x^3 - 6x^2 + 6x - 3$$

Applying Eq. (4.43)

$$\int_0^1 w_1 R\,dx = 0 \quad \rightarrow \quad 7a_1 - 2a_2 = 8$$
$$\int_0^1 w_2 R\,dx = 0 \quad \rightarrow \quad -112a_1 + 438a_2 = 161$$

Thus

$$a_1 = \frac{3826}{2842}, \qquad a_2 = \frac{2023}{2842}$$

and Eq. (4.55) becomes

$$\widetilde{\Phi}(x) = -1.6925x + 0.2785x^2 + 0.7118x^3 \tag{4.61}$$

Notice that the approximate solutions in Eqs. (4.57) to (4.61) all satisfy the boundary conditions $\Phi(0) = 0$ and $\Phi'(1) = 1$. The five approximate solutions are compared in Table 4.3. ∎

Table 4.3 Comparison of the Weighted Residual Solutions of the Problem in Example 4.7 with the Exact Solution in Eq. (4.52)

x	Exact solution	Collocation 1	Collocation 2	Subdomain	Galerkin	Least squares
0.0	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.1	−0.1736	−0.1435	−0.1602	−0.1753	−0.1794	−0.1657
0.2	−0.3402	−0.2772	−0.3108	−0.3403	−0.3451	−0.3217
0.3	−0.4928	−0.3975	−0.4477	−0.4907	−0.4935	−0.4635
0.4	−0.6248	−0.5006	−0.5666	−0.6221	−0.6208	−0.5869
0.5	−0.7303	−0.5827	−0.6633	−0.7300	−0.7233	−0.6877
0.6	−0.8042	−0.6400	−0.7336	−0.8100	−0.7972	−0.7615
0.7	−0.8424	−0.6690	−0.7734	−0.8577	−0.8390	−0.8041
0.8	−0.8422	−0.6657	−0.7785	−0.8687	−0.8449	−0.8113
0.9	−0.8019	−0.6264	−0.7446	−0.8385	−0.8111	−0.7788
1.0	−0.7216	−0.5476	−0.6676	−0.7627	−0.7340	−0.7022

Example 4.8

This example illustrates the fact that expansion functions u_n must be selected such that $L\,u_n$ form a complete set for the range of the operator L. Consider the differential equation

$$-\Phi'' = 2 + \sin x, \qquad 0 \le x \le 2\pi \tag{4.62}$$

subject to

$$\Phi(0) = \Phi(2\pi) = 0 \tag{4.63}$$

Suppose we carelessly select

$$u_n = \sin nx, \qquad n = 1, 2, \ldots \tag{4.64}$$

as the expansion functions, the approximate solution is

$$\widetilde{\Phi} = \sum_{n=1}^{N} a_n \sin nx \tag{4.65}$$

If we apply the Galerkin method, we obtain

$$\widetilde{\Phi} = \sin x \tag{4.66}$$

Although u_n satisfy both the differentiability and boundary conditions, Eq. (4.66) does not satisfy Eq. (4.62). Hence Eq. (4.66) is an incorrect solution. The problem is that the set $\{\sin nx\}$ does not form a complete set. If we add constant and cosine terms to the expansion functions in Eq. (4.65), then

$$\widetilde{\Phi} = a_0 + \sum_{n=1}^{N} [a_n \sin nx + b_n \cos nx] \tag{4.67}$$

As $N \to \infty$, Eq. (4.67) is the classical Fourier series solution. Applying the Galerkin method leads to

$$\widetilde{\Phi} = \sin nx \tag{4.68}$$

which is again an incorrect solution. The problem is that even though u_n form a complete set, $L\,u_n$ do not. For example, nonzero constants cannot be approximated by $L\,u_n$. In order for $L\,u_n$ to form a complete set, $\widetilde{\Phi}$ must be of the form

$$\widetilde{\Phi} = \sum_{n=1}^{n} [a_n \sin nx + b_n \cos nx] + a_0 + cx + dx^2 \tag{4.69}$$

Notice that the expansion functions $\{1, x, x^2, \sin nx, \cos nx\}$ in the interval $[0, 2\pi]$ form a linearly dependent set. This is because any function such as x or x^2 can be represented in the interval $[0, 2\pi]$ by the set $\{\sin nx, \cos nx\}$. Applying the Galerkin method, Eq. (4.69) leads to

$$\widetilde{\Phi} = \sin x + x(2\pi - x) \tag{4.70}$$

which is the exact solution Φ. □

4.7 Eigenvalue Problems

As mentioned in Section 1.3.2, eigenvalue (nondeterministic) problems are described by equations of the type

$$L\Phi = \lambda M\Phi \tag{4.71}$$

where L and M are differential or integral, scalar or vector operators. The problem here is the determination of the eigenvalues λ and the corresponding eigenfunctions

Φ. It can be shown [11] that the variational principle for λ takes the form

$$\boxed{\lambda = \min \frac{\langle \Phi, L\Phi \rangle}{\langle \Phi, M\Phi \rangle} = \min \frac{\int \Phi L \Phi \, dv}{\int \Phi M \Phi \, dv}} \tag{4.72}$$

We may apply Eq. (4.72) to the Helmholtz equation for scalar waves, for example,

$$\nabla^2 \Phi + k^2 \Phi = 0 \tag{4.73}$$

Comparing Eq. (4.73) with Eq. (4.71), we obtain $L = -\nabla^2$, $M = 1$ (the identity operator), $\lambda = k^2$ so that

$$k^2 = \min \frac{\int \Phi \nabla^2 \Phi \, dv}{\int \Phi^2 \, dv} \tag{4.74}$$

Applying Green's identity (see Example 1.1),

$$\int_v \left(U \nabla^2 V + \nabla U \cdot \nabla V \right) dv = \oint U \frac{\partial V}{\partial n} \, dS \, ,$$

to Eq. (4.74) yields

$$\boxed{k^2 = \min \frac{\int_v |\nabla \Phi|^2 \, dv - \oint \Phi \frac{\partial \Phi}{\partial n} \, dS}{\int_v \Phi^2 \, dv}} \tag{4.75}$$

Consider the following special cases.

(a) For homogeneous boundary conditions of the Dirichlet type ($\Phi = 0$) or Neumann type $\left(\dfrac{\partial \Phi}{\partial n} = 0 \right)$. Equation (4.75) reduces to

$$k^2 = \min \frac{\int_v |\nabla \Phi|^2 \, dv}{\int_v \Phi^2 \, dv} \tag{4.76}$$

(b) For mixed boundary conditions $\left(\dfrac{\partial \Phi}{\partial n} + h\Phi = 0 \right)$, Eq. (4.75) becomes

$$k^2 = \min \frac{\int_v |\nabla \Phi|^2 \, dv + \oint h \Phi^2 \, dS}{\int_v \Phi^2 \, dv} \tag{4.77}$$

It is usually possible to solve Eq. (4.71) in a different way. We choose the basis functions $u_1, u_2, \ldots, u_N$ which satisfy the boundary conditions and assume the approximate solution

$$\widetilde{\Phi} = a_1 u_1 + a_2 u_2 + \cdots + a_N u_N$$

or

$$\widetilde{\Phi} = \sum_{n=1}^{N} a_n u_n \tag{4.78}$$

Substituting this into Eq. (4.71) gives

$$\sum_{n=1}^{N} a_n L u_n = \lambda \sum_{n=1}^{N} a_n M u_n \tag{4.79}$$

Choosing the weighting functions w_m and taking the inner product of Eq. (4.79) with each w_m, we obtain

$$\sum_{n=1}^{N} \big[\langle w_m, L u_n \rangle - \lambda \langle w_m, M u_n \rangle \big] a_n = 0, \qquad m = 1, 2, \ldots, N \tag{4.80}$$

This can be cast into matrix form as

$$\sum_{n=1}^{N} (A_{mn} - \lambda B_{mn}) X_n = 0 \tag{4.81}$$

where $A_{mn} = \langle w_m, L u_n \rangle$, $B_{mn} = \langle w_m, M u_n \rangle$, $X_n = a_n$. Thus we have a set of homogeneous equations. In order for $\widetilde{\Phi}$ in Eq. (4.78) not to vanish, it is necessary that the expansion coefficients a_n not all be equal to zero. This implies that the determinant of simultaneous equations (4.81) vanish, i.e.,

$$\begin{vmatrix} A_{11} - \lambda B_{11} & A_{12} - \lambda B_{12} & \cdots & A_{1N} - \lambda B_{1N} \\ \vdots & & & \vdots \\ A_{N1} - \lambda B_{N1} & A_{N2} - \lambda B_{N2} & \cdots & A_{NN} - \lambda B_{NN} \end{vmatrix} = 0$$

or

$$|[A] - \lambda [B]| = 0 \tag{4.82}$$

Solving this gives N approximate eigenvalues $\lambda_1, \ldots, \lambda_N$. The various ways of choosing w_m leads to different weighted residual techniques as discussed in the previous section.

Examples of eigenvalue problems for which variational methods have been applied include [28]–[37]:

- the cutoff frequency of a waveguide,
- the propagation constant of a waveguide, and
- the resonant frequency of a resonator.

Example 4.9

Solve the eigenvalue problem

$$\Phi'' + \lambda\Phi = 0, \qquad 0 < x < 1$$

with boundary conditions $\Phi(0) = 0 = \Phi(1)$. □

Solution

The exact eigenvalues are

$$\lambda_n = (n\pi)^2, \qquad n = 1, 2, 3, \ldots \tag{4.83}$$

and the corresponding (normalized) eigenfunctions are

$$\Phi_n = \sqrt{2}\sin(n\pi x) \tag{4.84}$$

where Φ_n has been normalized to unity, i.e., $\langle \Phi_n, \Phi_n \rangle = 1$.

The approximate eigenvalues and eigenfunctions can be obtained by either using Eq. (4.72) or Eq. (4.82). Let the approximate solution be

$$\widetilde{\Phi}(x) = \sum_{k=0}^{N} a_k u_k, \qquad u_k = x\left(1 - x^k\right) \tag{4.85}$$

From the given problem, $L = -\dfrac{d^2}{dx^2}$, $M = 1$ (identity operator). Using the Galerkin method, $w_m = u_m$.

$$\begin{aligned} A_{mn} &= \langle u_m, Lu_n \rangle = \int_0^1 \left(x - x^{m+1}\right)\left[-\frac{d^2}{dx^2}\left(x - x^{n+1}\right)\right] dx \\ &= \frac{mn}{m+n+1}, \end{aligned} \tag{4.86}$$

$$\begin{aligned} B_{mn} &= \langle u_m, Mu_n \rangle = \int_0^1 \left(x - x^{m+1}\right)\left(x - x^{n+1}\right) dx \\ &= \frac{mn(m+n+6)}{3(m+3)(n+3)(m+n+3)} \end{aligned} \tag{4.87}$$

The eigenvalues are obtained from

$$|\,[A] - \lambda\,[B]\,| = 0 \tag{4.88}$$

For $N = 1$,

$$A_{11} = \frac{1}{3}, \qquad B_1 = \frac{1}{30},$$

giving

$$\frac{1}{3} - \lambda \frac{1}{30} = 0 \qquad \rightarrow \qquad \lambda = 10$$

The first approximate eigenvalue is $\lambda = 10$, a good approximation to the exact value of $\pi^2 = 9.8696$. The corresponding eigenfunction $\widetilde{\Phi} = a_1(x - x^2)$ can be normalized to unity so that

$$\widetilde{\Phi} = \sqrt{30}(x - x^2)$$

For $N = 2$, evaluating Eqs. (4.86) and (4.87), we obtain

$$\begin{bmatrix} \frac{1}{3} & \frac{1}{2} \\ \frac{1}{2} & \frac{4}{5} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \lambda \begin{bmatrix} \frac{1}{30} & \frac{1}{20} \\ \frac{1}{20} & \frac{8}{105} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$$

or

$$\begin{vmatrix} 10 - \lambda & 0 \\ 0 & 42 - \lambda \end{vmatrix} = 0$$

giving eigenvalues $\lambda_1 = 10$, $\lambda_2 = 42$, compared with the exact values $\lambda_1 = \pi^2 = 9.8696$, $\lambda_2 = 4\pi^2 = 39.4784$, and the corresponding normalized eigenfunctions are

$$\widetilde{\Phi}_1 = \sqrt{30}\left(x - x^2\right)$$
$$\widetilde{\Phi}_2 = 2\sqrt{210}\left(x - x^2\right) - 2\sqrt{210}\left(x - x^3\right)$$

Continuing this way for higher N, the approximate eigenvalues shown in Table 4.4 are obtained. Unfortunately, the labor of computation increases as more u_k are included in $\widetilde{\Phi}$. Notice from Table 4.4 that the approximate eigenvalues are always greater than the exact values. This is always true for a self-adjoint, positive definite operator [17]. Figure 4.3 shows the comparison between the approximate and exact eigenfunctions. ∎

Table 4.4 Comparison Between Approximate and Exact Eigenvalues for Example 4.9

Exact	Approximate			
	$N = 1$	$N = 2$	$N = 3$	$N = 4$
9.870	10.0	10.0	9.8697	9.8697
39.478		42.0	39.497	39.478
88.826			102.133	102.133
157.914				200.583

Example 4.10

Calculate the cutoff frequency of the inhomogeneous rectangular waveguide shown in Fig. 4.4. Take $\epsilon = 4\epsilon_o$ and $s = a/3$. □

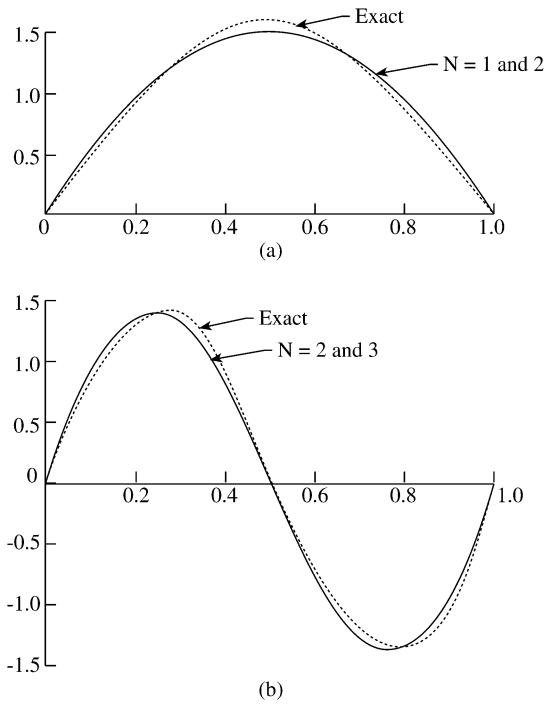

Figure 4.3
Comparison of approximate eigenfunctions with the exact solutions: (a) first eigenfunction, (b) second eigenfunction. (After Harrington [17]; with permission of Krieger Publishing Co.).

Solution

We will find the lowest mode having $\dfrac{\partial}{\partial y} \equiv 0$. It is this dominant mode that is of most practical value. Since the dielectric constant varies from one region to another, it is reasonable to choose Φ to be an electric field, i.e., $\Phi = E_y$. Also, since $k^2 = \dfrac{\omega^2}{u^2} = \omega^2 \mu \epsilon$, Eq. (4.74) becomes

$$\omega^2 \mu_o \epsilon_o \int_0^s E_y^2 \, dx + \omega^2 \mu_o \epsilon_o \epsilon_r \int_s^{a-s} E_y^2 \, dx + \omega^2 \mu_o \epsilon_o \int_{a-s}^{a} E_y^2 \, dx = -\int_0^a E_y \frac{d^2 E_y}{dx^2} \, dx \tag{4.89}$$

Notice that in this implementation of Eq. (4.74), there are no coefficients so that there is nothing to minimize. We simply take k^2 as a ratio. Equation (4.89) can be written

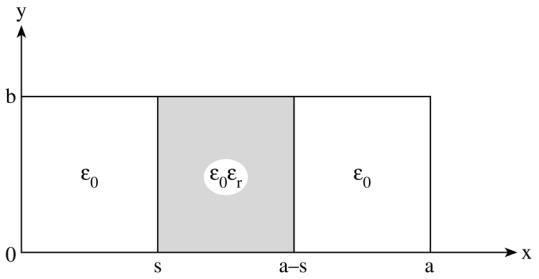

Figure 4.4
A symmetrically loaded rectangular waveguide.

as

$$\omega^2\mu_o\epsilon_o\int_0^a E_y^2\,dx + \omega^2\mu_o\epsilon_o(\epsilon_r-1)\int_s^{a-s} E_y^2\,dx = -\int_0^a E_y\frac{d^2E_y}{dx^2}\,dx \quad (4.90)$$

We now choose the trial function for E_y. It must be chosen to satisfy the boundary conditions, namely, $E_y = 0$ at $x = 0, a$. Since $E_y \sim \sin\dfrac{n\pi x}{a}$ for the empty waveguide, it makes sense to choose the trial function of the form

$$E_y = \sum_{n=1,3,5}^{\infty} c_n \sin\frac{n\pi x}{a} \quad (4.91)$$

We choose the odd values of n because the dielectric is symmetrically placed; otherwise we would have both odd and even terms.

Let us consider the trial function

$$E_y = \sin\frac{\pi x}{a} \quad (4.92)$$

Substituting Eq. (4.92) into Eq. (4.90) yields

$$\omega^2\mu_o\epsilon_o\int_0^a \sin^2\frac{\pi x}{a}\,dx + \omega^2\mu_o\epsilon_o(\epsilon_r-1)\int_s^{a-s}\sin^2\frac{\pi x}{a}\,dx = \frac{\pi^2}{a^2}\int_0^a \sin^2\frac{\pi x}{a}\,dx \quad (4.93)$$

which leads to

$$\omega^2\mu_o\epsilon_o\left\{1+(\epsilon_r-1)\left[\left(1-\frac{2s}{a}\right)+\frac{1}{\pi}\sin\frac{2\pi s}{a}\right]\right\} = \frac{\pi^2}{a^2}$$

But $k_o^2 = \omega^2\mu_o\epsilon_o = \dfrac{4\pi^2}{\lambda_c^2}$, where λ_c is the cutoff wavelength of the waveguide filled with vacuum. Hence

$$\frac{4\pi^2}{\lambda_c^2} = \frac{(\pi/a)^2}{1+(\epsilon_r-1)\left[(1-\frac{2s}{a})+\frac{1}{\pi}\sin\frac{2\pi s}{a}\right]}$$

Taking $\epsilon_r = 4$ and $s = a/3$ gives

$$\frac{4\pi^2}{\lambda_c^2} = \frac{(\pi/a)^2}{2 + \frac{3\sqrt{3}}{2\pi}}$$

or

$$\frac{a}{\lambda_c} = 0.2974$$

This is a considerable reduction in a/λ_c compared with the value of $a/\lambda_c = 0.5$ for the empty guide. The accuracy of the result may be improved by choosing more terms in Eq. (4.91). ∎

4.8 Practical Applications

The various techniques discussed in this chapter have been applied to solve a considerable number of EM problems. We select a simple example for illustration [38, 39]. This example illustrates the conventional use of the least squares method.

Consider a strip transmission line enclosed in a box containing a homogeneous medium as shown in Fig. 4.5. If a TEM mode of propagation is assumed, Laplace's

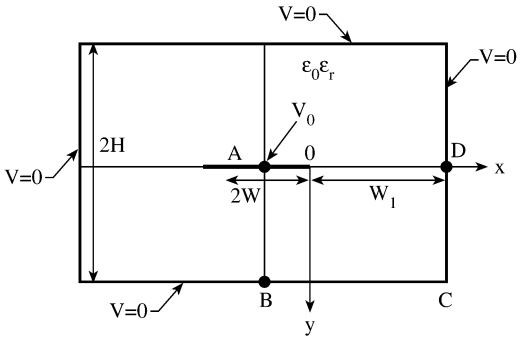

Figure 4.5
The strip line enclosed in a shielded box.

equation

$$\nabla^2 V = 0 \tag{4.94}$$

is obeyed. Due to symmetry, we will consider only one quarter section of the line as in Fig. 4.6 and adopt a boundary condition $\frac{\partial V}{\partial x} = 0$ at $x = -W$. We allow for the singularity at the edge of the strip. The variation of the potential in the vicinity of

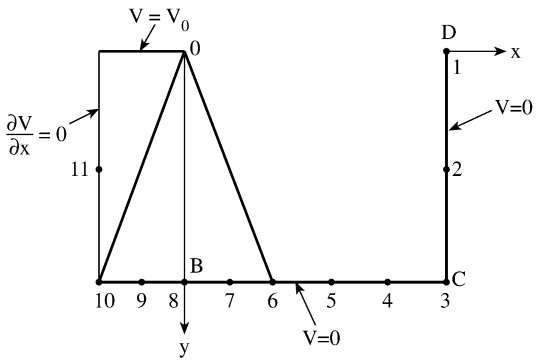

Figure 4.6
A quarter-section of the strip line.

such a singularity is approximated, in terms of trigonometric basis functions, as

$$V = V_o + \sum_{k=1,3,5}^{\infty} c_k \rho^{k/2} \cos \frac{k\phi}{2} , \tag{4.95}$$

where V_o is the potential on the trip conductor and the expansion coefficients c_k are to be determined. If we truncate the infinite series in Eq. (4.95) so that we have N unknown coefficients, we determine the coefficients by requiring that Eq. (4.95) be satisfied at $M(\geq N)$ points on the boundary. If $M = N$, we are applying the collocation method. If $M > N$, we obtain an overdetermined system of equations which can be solved by the method of least squares. Enforcing Eq. (4.95) at M boundary points, we obtain M simultaneous equations

$$\begin{bmatrix} V_1 \\ V_2 \\ \vdots \\ V_M \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & \dots & A_{1N} \\ A_{21} & A_{22} & \dots & A_{2N} \\ \vdots & & & \vdots \\ A_{M1} & A_{M2} & \dots & A_{MN} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_M \end{bmatrix}$$

i.e.,

$$[V] = [A][X] \tag{4.96}$$

where $[X]$ is an $N \times 1$ matrix containing the unknown expansion coefficients, $[V]$ is an $M \times 1$ column matrix containing the boundary conditions, and $[A]$ is the $M \times N$ coefficient matrix. Due to redundancy, $[X]$ cannot be uniquely determined from Eq. (4.96) if $M > N$. To solve this redundant system of equations by the method of least squares, we define the residual matrix $[R]$ as

$$[R] = [A][X] - [V] \tag{4.97}$$

We seek for $[X]$, which minimizes $[R]^2$. Consider

$$[I] = [R]^t\,[R] = [[A]\,[X] - [V]]^t\,[[A]\,[X] - [V]]$$
$$\frac{\partial\,[I]}{\partial\,[X]} = 0 \quad \rightarrow \quad [A]^t\,[A]\,[X] - [A]^t\,[V] = 0$$

or

$$[X] = \left[[A]^t\,[A]\right]^{-1}\,[A]^t\,[V] \tag{4.98}$$

where the superscript t denotes the transposition of the relevant matrix. Thus we have reduced the original redundant system of equations to a determinate set of N simultaneous equations in N unknown coefficients $c_1, c_2, \ldots, c_N$.

Once $[X] = [c_1, c_2, \ldots, c_N]$ is determined from Eq. (4.98), the approximate solution in Eq. (4.95) is completely determined. We can now determine the capacitance and consequently the characteristic impedance of the line for a given value of width-to-height ratio. The capacitance is determined from

$$C = \frac{Q}{V_o} = Q \tag{4.99}$$

If we let $V_o = 1$ V. The characteristic impedance is found from [40]

$$Z_o = \frac{\sqrt{\epsilon_r}}{cC} \tag{4.100}$$

where $c = 3 \times 10^8$ m/s, the speed of light in vacuum. The major problem here is finding Q in Eq. (4.99). If we divide the boundary BCD into segments,

$$Q = \int \rho_L\,dl = 4\sum_{BCD} \rho_L\,\Delta l$$

where the charge density $\rho_L = \mathbf{D}\cdot\mathbf{a}_n = \epsilon\mathbf{E}\cdot\mathbf{a}_n$, $\mathbf{E} = -\nabla V$, and the factor 4 is due to the fact that we consider only one quarter section of the line. But

$$\nabla V = \frac{\partial V}{\partial \rho}\mathbf{a}_\rho + \frac{1}{\rho}\frac{\partial V}{\partial \phi}\mathbf{a}_\phi\,,$$
$$\mathbf{E} = -\sum_{k=\text{odd}} \frac{k}{2} c_k \rho^{k/2-1}\left(\cos\frac{k\phi}{2}\mathbf{a}_\rho - \sin\frac{k\phi}{2}\mathbf{a}_\phi\right)$$

Since $\mathbf{a}_x = \cos\phi\mathbf{a}_\rho - \sin\phi\mathbf{a}_\phi$ and $\mathbf{a}_y = \sin\phi\mathbf{a}_\rho + \cos\phi\mathbf{a}_\phi$,

$$\rho_L|_{CD} = \epsilon\mathbf{E}\cdot\mathbf{a}_x$$
$$= -\epsilon\sum_{k=\text{odd}} \frac{k}{2} c_k \rho^{k/2-1}\left(\cos\frac{k\phi}{2}\cos\phi + \sin\frac{k\phi}{2}\sin\phi\right) \tag{4.101a}$$

and

$$\rho_L|_{BC} = \epsilon \mathbf{E} \cdot \mathbf{a}_y$$
$$= -\epsilon \sum_{k=\text{odd}} \frac{k}{2} c_k \rho^{k/2-1} \left(\cos \frac{k\phi}{2} \sin \phi - \sin \frac{k\phi}{2} \cos \phi \right) \tag{4.101b}$$

Example 4.11

Using the collocation (or point matching) method, write a computer program to calculate the characteristics impedance of the line shown in Fig. 4.5. Take:

(a) $W = H = 1.0$ m, $W_1 = 5.0$ m, $\epsilon_r = 1$, $V_0 = 1$ V,

(b) $W = H = 0.5$ m, $W_1 = 5.0$ m, $\epsilon_r = 1$, $V_0 = 1$ V. □

Solution

The computer program is presented in Fig. 4.7. For the first run, we take the number of matching points $N = 11$; the points are selected as illustrated in Fig. 4.6. The selection of the points is based on our prior knowledge of the fact that the flux lines are concentrated on the side of the strip line numbered 6 to 10; hence more points are chosen on that side.

The first step is to determine the potential distribution within the strip line using Eq. (4.95). In order to determine the expansion coefficients c_k in Eq. (4.95), we let Eq. (4.95) be satisfied at the matching points. On points 1 to 10 in Fig. 4.6, $V = 0$ so that Eq. (4.95) can be written as

$$-V_o = \sum_{k=1,3,5}^{\infty} c_k \rho^{k/2} \cos \frac{k\phi}{2} \tag{4.102}$$

The infinite series is terminated at $k = 19$ so that 10 points are selected on the sides of the strip line. The 11th point is selected such that $\dfrac{\partial V}{\partial x} = 0$ is satisfied at the point. Hence at point 11,

$$0 = \frac{\partial V}{\partial x} = \cos \phi \frac{\partial V}{\partial \rho} - \frac{\sin \phi}{\rho} \frac{\partial V}{\partial \phi}$$

or

$$0 = \sum_{k=1,3,5} \frac{k}{2} c_k \rho^{k/2-1} \left(\cos \frac{k\phi}{2} \cos \phi + \sin \frac{k\phi}{2} \sin \phi \right) \tag{4.103}$$

With Eqs. (4.102) and (4.103), we set up a matrix equation of the form

$$[B] = [F]\,[A] \tag{4.104}$$

```
0001    C  ******************************************************
0002    C  THIS PROGRAM CALCULATES THE CHARACTERISTIC IMPEDANCE
0003    C  Zo  OF A BOXED MICROSTRIP LINE USING COLLOCATION
0004    C  POINT-MATCHING METHOD
0005    C  ******************************************************
0006
0007          DIMENSION X(30), Y(30), R(30), PHI(30), F(30,30)
0008          DIMENSION A(30), B(30)
0009          DATA (X(I),I=1,11)/3*5.0,4.0,2.0,1.0,0.5,
                    0.0,-0.5,2*-1.0/
0010          DATA ( Y(J),J=1,11 )/0.0,0.5,8*1.0,0.5/
0011          DATA E0,PIE,VL/8.8541E-12,3.141592,3.0E+8/
0012
0013    C     INPUT DATA
0014           ER = 1.0
0015           EPS = E0*ER
0016           W = 1.0
0017           H = 1.0
0018           W1 = 5.0
0019           V0 = 1.0
0020           NMAX = 11
0021           NBC = 7     ! NO. OF POINTS ON BC OR X-AXIS
0022           NCD = NMAX - NBC  ! NO. OF POINTS ON DC OR Y-AXIS
0023           DX = (W + W1)/FLOAT(NBC)
0024           DY = H/FLOAT(NCD)
0025    C
0026    C      CALCULATE (R,PHI) FOR EACH POINT (X,Y)
0027    C
0028           DO 40 N = 1,NMAX
0029           R(N) = SQRT( X(N)**2 + Y(N)**2 )
0030           IF( X(N) ) 10,20,30
0031    10      PHI(N) = PIE - ATAN( Y(N)/ABS(X(N)) )
0032            GO TO 40
0033    20      PHI(N) = PIE/2.0
0034            GO TO 40
0035    30      PHI(N) = ATAN( Y(N)/X(N) )
0036    40      CONTINUE
0037    C
0038    C       CALCULATE MATRICES F(I,J) AND B(I)
0039    C
0040            DO 60 I = 1,NMAX
0041            B(I) = -V0
0042            M = 0
0043            DO 60 J = 1,NMAX
0044            M = 2*J - 1
0045            FM = FLOAT(M)/2.0
0046            IF(I.EQ.NMAX) GO TO 50
0047            F(I,J) = ( R(I)**(FM) )*COS(FM*PHI(I))
0048            GO TO 60
0049    50      CC = COS(PHI(I))*COS(PHI(I)*FM)
0050            SS = SIN(PHI(I))*SIN(PHI(I)*FM)
0051            F(I,J) = ( R(I)**(FM-1.) )*(CC + SS)*FM
0052            B(I) = 0.0
0053    60      CONTINUE
0054    C
0055    C       DETERMINE THE EXPANSION COEFFICIENTS A(I)
0056    C
0057            IDM = 30
0058            CALL INVERSE(F,NMAX,IDM)
0059            DO 80 I = 1, NMAX
```

Figure 4.7

Computer program for Example 4.11 ***(Continued)*****.**

```
0060            A(I) = 0.0
0061            DO 70 J=1,NMAX
0062            A(I) = A(I) + F(I,J)*B(J)
0063    70      CONTINUE
0064    80      CONTINUE
0065    C
0066    C       NOW, CALCULATE CHARGE ON THE X-SIDE, i.e. BC
0067    C
0068            RHO = 0.0
0069            YC = H
0070            XC = - W - DX/2.0
0071            DO 120 I=1,NBC
0072            XC = XC + DX
0073            RC = SQRT(XC**2 + YC**2 )
0074            IF(XC) 90,90,100
0075    90      PC = PIE - ATAN(YC/ABS(XC))
0076            GO TO 110
0077    100     PC = ATAN(YC/XC)
0078    110     CONTINUE
0079            DO 120 K=1,NMAX
0080            FK = FLOAT(2*K - 1)/2.0
0081            RRO = SIN(PC)*COS(FK*PC) - COS(PC)*SIN(FK*PC)
0082            RHO = RHO - A(K)*FK*(RC**(FK-1))*RRO*DX*EPS
0083    120     CONTINUE
0084    C
0085    C      NEXT, CALCULATE THE CHARGE ON THE Y-SIDE, i.e. DC
0086    C
0087            XC = W1
0088            YC = - DY/2.0
0089            DO 130 I=1,NCD
0090            YC = YC + DY
0091            RC = SQRT(XC**2 + YC**2)
0092            PC = ATAN(YC/XC)
0093            DO 130 K =1, NMAX
0094            FK = FLOAT(2*K - 1)/2.0
0095            RRO = COS(PC)*COS(FK*PC) + SIN(PC)*SIN(FK*PC)
0096            RHO = RHO - A(K)*FK*(RC**(FK-1))*RRO*DY*EPS
0097    130     CONTINUE
0098    C
0099    C       CALCULATE THE CHARACTERISTIC IMPEDANCE Zo
0100    C
0101            Q = 4.0*RHO
0102            C = ABS(Q)/V0
0103            ZO = SQRT(ER)/( C*VL )
0104            PRINT *,C,ZO
0105            WRITE(6,140) ZO
0106    140     FORMAT(2X,'Zo = ',3X,E12.6)
0107            STOP
0108            END
```

Figure 4.7
(Cont.) **Computer program for Example 4.11.**

where

$$B_k = \begin{cases} -V_o, & k \neq N \\ 0, & k = N \end{cases},$$

$$F_{ki} = \begin{cases} \rho_i^{k/2} \cos k\phi_i/2, & i = 1, \ldots, N-1, \\ & k = 1, \ldots, N \\ \dfrac{k}{2}\rho_i^{k/2-1} \left(\cos\left(k\phi_i/2\right)\cos\phi_i + \sin\left(k\phi_i/2\right)\sin\phi_i\right), & i = N,\ k = 1, \ldots, N \end{cases}$$

where (ρ_i, ϕ_i) are the cylindrical coordinates of the ith point. Matrix $[A]$ consists of the unknown expansion coefficients c_k. By matrix inversion using subroutine INVERSE in Appendix D, we obtain $[A]$ as

$$[A] = [F]^{-1}[B] \tag{4.105}$$

Once the expansion coefficients are determined, we now calculate the total charge on the sides of the strip line using Eq. (4.101) and

$$Q = 4 \sum_{BDC} \rho_L \Delta l$$

Finally, we obtain Z_o from Eqs. (4.99) and (4.100). Table 4.5 shows the results obtained using the program in Fig. 4.7 for different cases. In Table 4.5, $N = N_x + N_y$, where N_x and N_y are the number of matching points selected along the x and y axes, respectively. By comparing Fig. 4.5 with Fig. 2.13, one may be tempted to apply Eq. (2.223) to obtain the exact solution of part (a) as 61.1 Ω. But we must recall that Eq. (2.223) was derived based on the assumption that $w >> b$ in Fig. 2.12 or $W >> H$ in Fig. 4.5. The assumption is not valid in this case, the exact solution given in [39] is more appropriate. ∎

Table 4.5 Characteristic Impedance of the Strip Transmission Line of Fig. 4.5; for Example 4.11 with $W_1 = 5.0$

$W = H$	N	N_x	N_y	c_1	Calculated $Z_o(\Omega)$	Exact [39] $Z_o(\Omega)$
1.0	7	5	2	−1.1549	67.846	65.16
	11	8	3	−1.1266	65.16	
0.5	7	5	2	−1.1549	96.92	100.57
	11	8	3	−1.1266	99.60	

4.9 Concluding Remarks

This chapter has provided an elementary introduction to the basic idea of variational techniques. The variational methods provide simple but powerful solutions to physical problems provided we can find approximate basis functions. A prominent feature of the variational method lies in the ability to achieve high accuracy with few terms in the approximate solution. A major drawback is the difficulty encountered in selecting the basis functions. In spite of the drawback, the variational methods have been very useful and provide basis for both the method of moments and the finite element method to be discussed in the forthcoming chapters.

Needless to say, our discussion on variational techniques in this chapter has only been introductory. An exhaustive treatment of the subject can be found in [1, 6, 10, 11], [41]–[43]. Various applications of variational methods to EM-related problems include:

- waveguides and resonators [28]–[37]
- transmission lines [38, 39], [44]–[47]
- acoustic radiation [48]
- wave propagation [49]–[51]
- transient problems [52]
- scattering problems [53]–[59].

The problem of variational principles for EM waves in inhomogeneous media is discussed in [60].

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Problems

4.1 Find $\langle u, v\rangle$ if:

(a) $u = x^2,\ v = 2 - x$ in the interval $-1 < x < 1$,

(b) $u = 1,\ v = x^2 - 2y^2$ in the rectangular region $0 < x < 1,\ 1 < y < 2$,

(c) $u = x + y,\ v = xz$ in the cylindrical region $x^2 + y^2 \le 4,\ 0 < z < 5$.

4.2 Show that:

(a) $\langle h(x), f(x)\rangle = \langle h(x), f(-x)\rangle$,

(b) $\langle h(ax), f(x)\rangle = \left\langle h(x), \frac{1}{a} f\left(\frac{x}{a}\right)\right\rangle$,

(c) $\left\langle \frac{df}{dx}, h(x)\right\rangle = -\left\langle f(x), \frac{dh}{dx}\right\rangle$,

(d) $\left\langle \frac{d^n f}{dx^n}, h(x)\right\rangle = (-1)^n - \left\langle f(x), \frac{d^n h}{dx^n}\right\rangle$

Note from (d) that $L = \dfrac{d}{dx}, \dfrac{d^3}{dx^3}$, etc., are not self-adjoint, whereas $L = \dfrac{d^2}{dx^2}, \dfrac{d^4}{dx^4}$, etc., are.

4.3 Find the Euler partial differential equation for each of the following functionals:

(a) $\displaystyle\int_a^b \sqrt{1 + y'}\, dx$

(b) $\displaystyle\int_a^b y\sqrt{1 + y'^2}\, dx$

(c) $\displaystyle\int_a^b \cos(xy')\, dx$

4.4 Repeat Problem 4.3 for the following functionals:

(a) $\displaystyle\int_a^b (y'^2 - y^2)\, dx$

(b) $\displaystyle\int_a^b [5y^2 - (y'')^2 + 10x]\, dx$

(c) $\displaystyle\int_a^b (3uv - u^2 + u'^2 - v'^2)\, dx$

4.5 Determine the extremal $y(x)$ for each of the following variational problems:

(a) $\displaystyle\int_0^1 (2y'^2 + yy' + y' + y)\, dx,\ y(0) = 0,\ y(1) = 1$

(b) $\displaystyle\int (y'^2 - y^2)\, dx,\ y(0) = 1,\ y(\pi/2)$

4.6 If L is a positive definite, self-adjoint operator and $L\Phi = g$ has a solution Φ_o, show that the function

$$I = \langle L\Phi, \Phi\rangle - 2\langle \Phi, g\rangle\ ,$$

where Φ and g are real functions, is minimized by the solution Φ_o.

4.7 Show that a function that minimizes the functional

$$I(\Phi) = \frac{1}{2}\int_S \left[|\nabla\Phi|^2 - k^2\Phi^2 + 2g\Phi\right] dS$$

is the solution to the inhomogeneous Helmholtz equation

$$\nabla^2\Phi + k^2\Phi = g$$

4.8 Show that minimizing the energy functional

$$I = \frac{1}{2}\int_v |\nabla V|^2\, dv$$

is equivalent to solving Laplace's equation.

4.9 Using Euler's equation, obtain the differential equation corresponding to the electrostatic field functional

$$I = \int_v \left[\frac{1}{2}\epsilon E^2 - \rho_v V\right] dv$$

where $E = |\mathbf{E}|$ and ρ_v is the volume charge density.

4.10 Repeat Problem 4.9 for the energy function for steady state currents

$$I = \int_v \frac{1}{2}\mathbf{J}\cdot\mathbf{E}\, dv$$

where $\mathbf{J} = \sigma\mathbf{E}$.

4.11 Poisson's equation in an anisotropic medium is

$$\frac{\partial}{\partial x}\left(\epsilon_x \frac{\partial V}{\partial x}\right) + \frac{\partial}{\partial y}\left(\epsilon_y \frac{\partial V}{\partial y}\right) + \frac{\partial}{\partial z}\left(\epsilon_z \frac{\partial V}{\partial z}\right) = -\rho_v$$

in three dimensions. Derive the functional for the boundary value problem. Assume ϵ_x, ϵ_y, and ϵ_z are constants.

4.12 Show that the variational principle for the boundary value problem

$$\nabla^2\Phi = f(x, y, z)$$

subject to the mixed boundary condition

$$\frac{\partial\Phi}{\partial n} + g\Phi = h \quad \text{on } S$$

is

$$I(\Phi) = \int_v \left[|\nabla\Phi|^2 - 2fg\right] dv + \oint \left[g\Phi^2 - 2h\Phi\right] dS$$

4.13 Obtain the variational principle for the differential equation

$$-\frac{d^2y}{dx^2} + y = \sin\pi x\,, \quad 0 < x < 1$$

subject to $y(0) = 0 = y(1)$.

4.14 Determine the variational principle for

$$\Phi'' = \Phi - 4xe^x \ , \quad 0 < x < 1$$

subject to $\Phi'(0) = \Phi(0) + 1$, $\Phi'(1) = \Phi(1) - e$.

4.15 For the boundary value problem

$$-\Phi'' = x \ , \ 0 < x < 1$$
$$\Phi(0) = 0 \ , \ \Phi(1) = 2$$

determine the approximate solution using the Rayleigh-Ritz method with basis functions

$$u_k = x^k(x-1) \ , \qquad k = 0, 1, 2, \ldots, M$$

Try cases when $M = 1, 2$, and 3.

4.16 Rework Example 4.5 using

(a) $u_m = x(1 - x^m)$,

(b) $u_m = \sin m\pi x$, $m = 1, 2, 3, \ldots, M$. Try cases when $M = 1, 2$, and 3.

4.17 Solve the differential equation

$$-\Phi''(x) + 0.1\Phi(x) = 1, \quad 0 \le x \le 10$$

subject to the boundary conditions $\Phi'(0) = 0 = \Phi(0)$ using the trial function

$$\widetilde{\Phi}(x) = a_1 \cos\frac{\pi x}{20} + a_2 \cos\frac{3\pi x}{20} + a_3 \cos\frac{5\pi x}{20}$$

Determine the expansion coefficients using: (a) collocation method, (b) subdomain method, (c) Galerkin method, (d) least squares method.

4.18 For the boundary value problem

$$\Phi'' + \Phi + x = 0 \ , \quad 0 < x < 1$$

with homogeneous boundary conditions $\Phi = 0$ at $x = 0$ and $x = 1$, determine the coefficients of the approximate solution function

$$\widetilde{\Phi}(x) = x(1-x)\,(a_1 + a_2 x)$$

using: (a) collocation method (choose $x = 1/4$, $x = 1/2$ as collocation points), (b) Galerkin method, (c) least squares method.

4.19 Given the boundary value problem

$$y'' + (1 + x^2)y + 1 = 0 \ , \quad -1 < x < 1 \ ,$$
$$y(-1) = 0 = y(1) \ ,$$

find the expansion coefficients of the approximate solution

$$\widetilde{y} = a_1(1 - x^2) + a_2 x^2(1 - x^2)$$

by using: (i) the various weighted residual methods, (ii) the Rayleigh-Ritz method.

4.20 Rework the previous problem using the approximate solution

$$\widetilde{y} = a_1(1 - x^2)(1 - 4x^2) + a_2 x^2(1 - x^2)$$

Use the Galerkin and the least squares methods to determine a_1 and a_2.

4.21 Consider the problem

$$\Phi'' + x\Phi' + \Phi = 2x\ , \quad 0 < x < 1$$

subject to $\Phi(0) = 1, \Phi(1) = 0$. Find the approximate solution using the Galerkin method. Use $u_k = x^k(1 - x)$, $k = 0, 1, \ldots, N$. Try $N = 3$.

4.22 Determine the first three eigenvalues of the equation

$$y'' + \lambda y = 0\ , \quad 0 < x < 1\ ,$$

$y(0) = 0 = y(1)$ using collocation at $x = 1/4, 1/2, 3/4$.

4.23 Determine the fundamental eigenvalue of the problem

$$-\Phi''(x) + 0.1\Phi(x) = \lambda\Phi(x)\ , \quad 0 < x < 10$$

subject to $\Phi(0) = 0 = \Phi(10)$. Use the trial function

$$\widetilde{\Phi}(x) = x(x - 10)$$

4.24 Obtain the lowest eigenvalue of the problem

$$\nabla^2\Phi + \lambda\Phi = 0\ , \quad 0 < \rho < 1$$

with $\Phi = 0$ at $\rho = 1$.

4.25 Rework Example 4.10 using the trial function

$$E_y = \sin\frac{\pi x}{a} + c_1 \sin\frac{3\pi x}{a}$$

where c_1 is a coefficient to be chosen such that $\omega^2\epsilon_o\mu_o$ is minimized.

4.26 Consider the waveguide in Fig. 4.4 as homogeneous. To determine the cutoff frequency, we may use the polynomial trial function

$$H_z = Ax^3 + Bx^2 + Cx + D$$

By applying the conditions

$$H_z = 1 \quad \text{at } x = a\,, \qquad H_z = -1 \text{ at } x = a\,,$$
$$\frac{\partial H_z}{\partial x} = 0 \quad \text{at } x = 0, a\,,$$

determine A, B, C, and D. Using the trial function, calculate the cutoff frequency.