Appendix A

Vector Relations

A.1 Vector Identities

If **A** and **B** are vector fields while U and V are scalar fields, then

$$\begin{aligned}
\nabla V(U+V) &= \nabla U + \nabla V \\
\nabla(UV) &= U\nabla V + V\nabla U \\
\nabla\left(\frac{U}{V}\right) &= \frac{\nabla(\nabla U) - U(\nabla V)}{V^2} \\
\nabla V^n &= nV^{n-1}\nabla V \qquad (n = \text{ integer}) \\
\nabla(\mathbf{A}\cdot\mathbf{B}) &= (A\cdot\nabla)\mathbf{B} + (\mathbf{B}\cdot\nabla)\mathbf{A} + \mathbf{A}\times(\nabla\times\mathbf{B}) + \mathbf{B}\times(\nabla\times\mathbf{A}) \\
\nabla\cdot(\mathbf{A}+\mathbf{B}) &= \nabla\cdot\mathbf{A} + \nabla\cdot\mathbf{B} \\
\nabla\cdot(\mathbf{A}\times\mathbf{B}) &= \mathbf{B}\cdot(\nabla\times\mathbf{A}) - \mathbf{A}\cdot(\nabla\times\mathbf{B}) \\
\nabla\cdot(\nabla\mathbf{A}) &= V\nabla\cdot\mathbf{A} + \mathbf{A}\cdot\nabla V \\
\nabla\cdot(\nabla V) &= \nabla^2 V \\
\nabla\cdot(\nabla\times\mathbf{A}) &= 0 \\
\nabla\times(\mathbf{A}+\mathbf{B}) &= \nabla\times\mathbf{A} + \nabla\times\mathbf{B} \\
\nabla\times(\mathbf{A}\times\mathbf{B}) &= \mathbf{A}(\nabla\cdot\mathbf{B}) - \mathbf{B}(\nabla\cdot\mathbf{A}) + (\mathbf{B}\cdot\nabla)\mathbf{A} - (\mathbf{A}\cdot\nabla)\mathbf{B} \\
\nabla\times(V\mathbf{A}) &= \nabla V\times\mathbf{A} + V(\nabla\times\mathbf{A}) \\
\nabla\times(\nabla V) &= 0 \\
\nabla\times(\nabla\times\mathbf{A}) &= \nabla(\nabla\cdot\mathbf{A}) - \nabla^2\mathbf{A}
\end{aligned}$$

A.2 Vector Theorems

If v is the volume bounded by the closed surface S, and $\mathbf{a}_n$ is a unit normal to S, then

$$\oint_S \mathbf{A}\cdot d\mathbf{S} = \int_v \nabla\cdot\mathbf{A}\,dv \quad \text{(Divergence theorem)}$$

$$\oint_S V\,d\mathbf{S} = \int_v \nabla V\,dv \quad \text{(Gradient theorem)}$$

$$\oint_S \mathbf{A}\times d\mathbf{S} = -\int_v \nabla\times\mathbf{A}\,dv$$

$$\oint_S \left[(\mathbf{A}\cdot d\mathbf{S})\mathbf{A} - \frac{1}{2}A^2\,d\mathbf{S}\right] = \int_v [(\nabla\times\mathbf{A})\times\mathbf{A} + \mathbf{A}\nabla\cdot\mathbf{A}]\,dv$$

$$\oint_S U\nabla V\cdot d\mathbf{S} = \int_v \left[U\nabla^2 V + \nabla U\cdot\nabla V\right]dv \quad \text{(Green 1st identity)}$$

$$\oint_S [U\nabla V - V\nabla U]\cdot d\mathbf{S} = \int_v \left[U\nabla^2 V - V\nabla^2 U\right]dv \quad \text{(Green 2nd identity)}$$

where $d\mathbf{S} = dS\,\mathbf{a}_n$.

If S is the area bounded by the closed path L and the positive directions of elements $d\mathbf{S}$ and $d\mathbf{l}$ are related by the right-hand rule, then

$$\oint_L \mathbf{A}\cdot d\mathbf{l} = \int_S \nabla\times\mathbf{A}\cdot d\mathbf{S} \quad \text{(Stokes' theorem)}$$

$$\oint_L V\,d\mathbf{l} = -\int_S \nabla V\times d\mathbf{S}$$

A.3 Orthogonal Coordinates

Rectangular Coordinates (x, y, z)

$$\nabla V = \frac{\partial V}{\partial x}\mathbf{a}_x + \frac{\partial V}{\partial y}\mathbf{a}_y + \frac{\partial V}{\partial z}\mathbf{a}_z$$

$$\nabla\cdot\mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$$\nabla\times\mathbf{A} = \left[\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right]\mathbf{a}_x + \left[\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right]\mathbf{a}_y + \left[\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right]\mathbf{a}_z$$

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}$$

$$\nabla^2\mathbf{A} = \nabla^2 A_x\mathbf{a}_x + \nabla^2 A_y\mathbf{a}_y + \nabla^2 A_z\mathbf{a}_z$$

Cylindrical Coordinates (ρ, ϕ, z)

$$\nabla V = \frac{\partial V}{\partial \rho}\mathbf{a}_\rho + \frac{1}{\rho}\frac{\partial V}{\partial \phi}\mathbf{a}_\phi + \frac{\partial V}{\partial z}\mathbf{a}_z$$

$$\nabla \cdot \mathbf{A} = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho A_\rho) + \frac{1}{\rho}\frac{\partial}{\partial \phi}A_\phi + \frac{\partial}{\partial z}A_z$$

$$\nabla \times \mathbf{A} = \left[\frac{1}{\rho}\frac{\partial}{\partial \phi}A_z - \frac{\partial}{\partial z}A_\phi\right]\mathbf{a}_\rho + \left[\frac{\partial}{\partial z}A_\rho - \frac{\partial}{\partial \rho}A_z\right]\mathbf{a}_\phi + \frac{1}{\rho}\left[\frac{\partial}{\partial \rho}(\rho A_\phi) - \frac{\partial}{\partial \phi}A_\rho\right]\mathbf{a}_z$$

$$\nabla^2 V = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial V}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2}$$

$$\nabla^2 \mathbf{A} = \left[\nabla^2 A_\rho - \frac{2}{\rho^2}\frac{\partial}{\partial \phi}A_\phi - \frac{A_\rho}{\rho^2}\right]\mathbf{a}_\rho + \left[\nabla^2 A_\phi + \frac{2}{\rho^2}\frac{\partial}{\partial \phi}A_\rho - \frac{A_\phi}{\rho^2}\right]\mathbf{a}_\phi + \nabla^2 A_z \mathbf{a}_z$$

Spherical Coordinates (r, θ, ϕ)

$$\nabla V = \frac{\partial V}{\partial r}\mathbf{a}_r + \frac{1}{r}\frac{\partial V}{\partial \theta}\mathbf{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\mathbf{a}_\phi$$

$$\nabla \cdot \mathbf{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta A_\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial \phi}$$

$$\nabla \times \mathbf{A} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial \theta}(\sin\theta A_\phi) - \frac{\partial}{\partial \phi}A_\theta\right]\mathbf{a}_r + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial \phi}A_r - \frac{\partial}{\partial r}(rA_\phi)\right]\mathbf{a}_\theta + \frac{1}{r}\left[\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial}{\partial \theta}A_r\right]\mathbf{a}_\phi$$

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial V}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial \phi^2}$$

$$\begin{aligned}\nabla^2 \mathbf{A} = {} & \left[\nabla^2 A_r - \frac{2}{r^2\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta A_\theta) - \frac{2}{r^2\sin\theta}\frac{\partial}{\partial \phi}A_\phi - \frac{2}{r^2}A_r\right]\mathbf{a}_r \\ & + \left[\nabla^2 A_\theta + \frac{2}{r^2}\frac{\partial}{\partial \theta}A_r - \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial}{\partial \phi}A_\phi - \frac{1}{r^2\sin^2\theta}A_\theta\right]\mathbf{a}_\theta \\ & + \left[\nabla^2 A_\phi + \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial}{\partial \phi}A_\theta + \frac{2}{r^2\sin\theta}\frac{\partial}{\partial \phi}A_r - \frac{1}{r^2\sin^2\theta}A_\phi\right]\mathbf{a}_\phi\end{aligned}$$