Chapter 2

Analytical Methods

"I've learned that about 90 percent of the things that happen to me are good and only about 10 percent are bad. To be happy, I just have to focus on the 90 percent."
Anonymous

2.1 Introduction

The most satisfactory solution of a field problem is an exact mathematical one. Although in many practical cases such an analytical solution cannot be obtained and we must resort to numerical approximate solution, analytical solution is useful in checking solutions obtained from numerical methods. Also, one would hardly appreciate the need for numerical methods without first seeing the limitations of the classical analytical methods. Hence our objective in this chapter is to briefly examine the common analytical methods and thereby put numerical methods in proper perspective.

The most commonly used analytical methods in solving EM-related problems include:

(1) separation of variables

(2) series expansion

(3) conformal mapping

(4) integral methods

Perhaps the most powerful analytical method is the separation of variables; it is the method that will be emphasized in this chapter. Since the application of conformal mapping is restricted to certain EM problems, it will not be discussed here. The interested reader is referred to Gibbs [1]. The integral methods will be covered in Chapter 5, and fully discussed in [2].

2.2 Separation of Variables

The method of separation of variables (sometimes called the method of Fourier) is a convenient method for solving a partial differential equation (PDE). Basically, it entails seeking a solution which breaks up into a product of functions, each of which involves only one of the variables. For example, if we are seeking a solution $\Phi(x, y, z, t)$ to some PDE, we require that it has the product form

$$\Phi(x, y, z, t) = X(x)Y(y)Z(z)T(t) \tag{2.1}$$

A solution of the form in Eq. (2.1) is said to be separable in x, y, z, and t. For example, consider the functions

(1) $x^2 yz \sin 10t$,
(2) $xy^2 + \frac{2}{t}$,
(3) $(2x + y^2)z \cos 10t$.

(1) is completely separable, (2) is not separable, while (3) is separable only in z and t.

To determine whether the method of independent separation of variables can be applied to a given physical problem, we must consider the PDE describing the problem, the shape of the solution region, and the boundary conditions — the three elements that uniquely define a problem. For example, to apply the method to a problem involving two variables x and y (or ρ and ϕ, etc.), three things must be considered [3]:

(i) The differential operator L must be separable, i.e., it must be a function of $\Phi(x, y)$ such that
$$\frac{L\{X(x)Y(y)\}}{\Phi(x, y)X(x)Y(y)}$$
is a sum of a function of x only and a function of y only.

(ii) All initial and boundary conditions must be on constant-coordinate surfaces, i.e., $x =$ constant, $y =$ constant.

(iii) The linear operators defining the boundary conditions at $x =$ constant (or $y =$ constant) must involve no partial derivatives of Φ with respect to y (or x), and their coefficient must be independent of y (or x).

For example, the operator equation

$$L\Phi = \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial x \partial y} + \frac{\partial^2 \Phi}{\partial y^2}$$

violates (i). If the solution region R is not a rectangle with sides parallel to the x and y axes, (ii) is violated. With a boundary condition $\Phi = 0$ on a part of $x = 0$ and $\partial\Phi/\partial x = 0$ on another part, (iii) is violated.

With this preliminary discussion, we will now apply the method of separation of variables to PDEs in rectangular, circular cylindrical, and spherical coordinate systems. In each of these applications, we shall always take these three major steps:

(1) separate the (independent) variables

(2) find particular solutions of the separated equations, which satisfy some of the boundary conditions

(3) combine these solutions to satisfy the remaining boundary conditions

We begin the application of separation of variables by finding the product solution of the homogeneous scalar wave equation

$$\nabla^2 \Phi - \frac{1}{c^2}\frac{\partial^2 \Phi}{\partial t^2} = 0 \tag{2.2}$$

Solution to Laplace's equation can be derived as a special case of the wave equation. Diffusion and heat equation can be handled in the same manner as we will treat wave equation. To solve Eq. (2.2), it is expedient that we first separate the time dependence. We let

$$\Phi(\mathbf{r}, t) = U(\mathbf{r})T(t) \tag{2.3}$$

Substituting this in Eq. (2.2),

$$T\nabla^2 U - \frac{1}{c^2}UT'' = 0$$

Dividing by UT gives

$$\frac{\nabla^2 U}{U} = \frac{T''}{c^2 T} \tag{2.4}$$

The left side is independent of t, while the right side is independent of $\mathbf{r}$; the equality can be true only if each side is independent of both variables. If we let an arbitrary constant $-k^2$ be the common value of the two sides, Eq. (2.4) reduces to

$$T'' + c^2k^2T = 0\,, \tag{2.5a}$$

$$\nabla^2 U + k^2 U = 0 \tag{2.5b}$$

Thus we have been able to separate the space variable $\mathbf{r}$ from the time variable t. The arbitrary constant $-k^2$ introduced in the course of the separation of variables is called the *separation constant*. We shall see that in general the total number of independent separation constants in a given problem is one less than the number of independent variables involved.

Equation (2.5a) is an ordinary differential equation with solution

$$T(t) = a_1 e^{jckt} + a_2 e^{-jckt} \tag{2.6a}$$

or

$$T(t) = b_1 \cos(ckt) + b_2 \sin(ckt) \tag{2.6b}$$

Since the time dependence does not change with a coordinate system, the time dependence expressed in Eq. (2.6) is the same for all coordinate systems. Therefore, we shall henceforth restrict our effort to seeking solution to Eq. (2.5b). Notice that if $k = 0$, the time dependence disappears and Eq. (2.5b) becomes Laplace's equation.

2.3 Separation of Variables in Rectangular Coordinates

In order not to complicate things, we shall first consider Laplace's equation in two dimensions and later extend the idea to wave equations in three dimensions.

2.3.1 Laplace's Equations

Consider the Dirichlet problem of an infinitely long rectangular conducting trough whose cross section is shown in Fig. 2.1. For simplicity, let three of its sides be

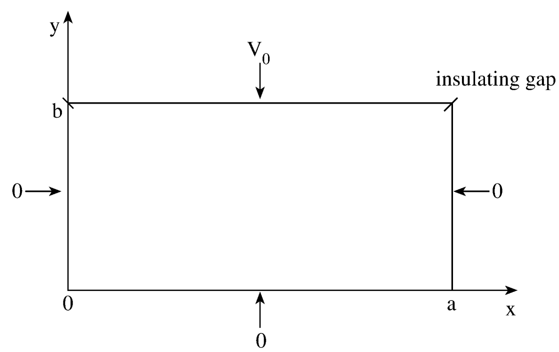


Figure 2.1
Cross section of the rectangular conducting trough.

maintained at zero potential while the fourth side is at a fixed potential V_o. This is a boundary value problem. The PDE to be solved is

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0 \tag{2.7}$$

subject to (Dirichlet) boundary conditions

$$V(0, y) = 0 \tag{2.8a}$$
$$V(a, y) = 0 \tag{2.8b}$$
$$V(x, 0) = 0 \tag{2.8c}$$
$$V(x, b) = V_o \tag{2.8d}$$

We let

$$V(x, y) = X(x)Y(y) \tag{2.9}$$

Substitute this into Eq. (2.7) and divide by XY. This leads to

$$\frac{X''}{X} + \frac{Y''}{Y} = 0$$

or

$$\frac{X''}{X} = -\frac{Y''}{Y} = \lambda \tag{2.10}$$

where λ is the separation constant. Thus the separated equations are

$$X'' - \lambda X = 0 \tag{2.11}$$

$$Y'' + \lambda Y = 0 \tag{2.12}$$

To solve the ordinary differential equations (2.11) and (2.12), we must impose the boundary conditions in Eq. (2.8). However, these boundary conditions must be transformed so that they can be applied directly to the separated equations. Since $V = XY$,

$$V(0, y) = 0 \quad \to \quad X(0) = 0 \tag{2.13a}$$

$$V(a, y) = 0 \quad \to \quad X(a) = 0 \tag{2.13b}$$

$$V(x, 0) = 0 \quad \to \quad Y(0) = 0 \tag{2.13c}$$

$$V(x, b) = V_o \quad \to \quad X(x)Y(b) = V_o \tag{2.13d}$$

Notice that only the homogeneous conditions are separable. To solve Eq. (2.11), we distinguish the three possible cases: $\lambda = 0$, $\lambda > 0$, and $\lambda < 0$.

Case 1: If $\lambda = 0$, Eq. (2.11) reduces to

$$X'' = 0 \qquad \text{or} \qquad \frac{d^2X}{dx^2} = 0 \tag{2.14}$$

which has the solution

$$X(x) = a_1 x + a_2 \tag{2.15}$$

where a_1 and a_2 are constants. Imposing the conditions in Eq. (2.13a) and Eq. (2.13b),

$$X(0) = 0 \quad \to \quad a_2 = 0$$

$$X(a) = 0 \quad \to \quad a_1 = 0$$

Hence $X(x) = 0$, a trivial solution. This renders case $\lambda = 0$ as unacceptable.

Case 2: If $\lambda > 0$, say $\lambda = \alpha^2$, Eq. (2.11) becomes

$$X'' - \alpha^2 X = 0 \tag{2.16}$$

with the corresponding auxiliary equations $m^2 - \alpha^2 = 0$ or $m = \pm\alpha$. Hence the general solution is

$$X = b_1 e^{-\alpha x} + b_2 e^{\alpha x} \tag{2.17}$$

or

$$X = b_3 \sinh \alpha x + b_4 \cosh \alpha x \tag{2.18}$$

The boundary conditions are applied to determine b_3 and b_4.

$$\begin{aligned} X(0) = 0 \qquad &\rightarrow \qquad b_4 = 0 \\ X(a) = 0 \qquad &\rightarrow \qquad b_3 = 0 \end{aligned}$$

since $\sinh \alpha x$ is never zero for $\alpha > 0$. Hence $X(x) = 0$, a trivial solution, and we conclude that case $\lambda > 0$ is not valid.

Case 3: If $\lambda < 0$, say $\lambda = -\beta^2$,

$$X'' + \beta^2 X = 0 \tag{2.19}$$

with the auxiliary equation $m^2 + \beta^2 = 0$ or $m = \pm j\beta$. The solution to Eq. (2.19) is

$$X = A_1 e^{j\beta x} + A_2 e^{j\beta x} \tag{2.20a}$$

or

$$X = B_1 \sin \beta x + B_2 \cos \beta x \tag{2.20b}$$

Again,

$$\begin{aligned} X(0) = 0 \qquad &\rightarrow \qquad B_2 = 0 \\ X(a) = 0 \qquad &\rightarrow \qquad \sin \beta a = 0 = \sin n\pi \end{aligned}$$

or

$$\beta = \frac{n\pi}{a}, \qquad n = 1, 2, 3, \dots \tag{2.21}$$

since B_1 cannot vanish for nontrivial solutions, whereas $\sin \beta a$ can vanish without its argument being zero. Thus we have found an infinite set of discrete values of λ for which Eq. (2.11) has nontrivial solutions, i.e.,

$$\lambda = -\beta^2 = \frac{-n^2\pi^2}{a^2}, \qquad n = 1, 2, 3, \dots \tag{2.22}$$

These are the eigenvalues of the problem and the corresponding eigenfunctions are

$$X_n(x) = \sin \beta x = \sin \frac{n\pi x}{a} \tag{2.23}$$

From Eq. (2.22) note that it is not necessary to include negative values of n since they lead to the same set of eigenvalues. Also we exclude $n = 0$ since it yields the trivial solution $X = 0$ as shown under Case 1 when $\lambda = 0$. Having determined λ, we can solve Eq. (2.12) to find $Y_n(y)$ corresponding to $X_n(x)$. That is, we solve

$$Y'' - \beta^2 Y = 0 \,, \tag{2.24}$$

which is similar to Eq. (2.16), whose solution is in Eq. (2.18). Hence the solution to Eq. (2.24) has the form

$$Y_n(y) = a_n \sinh \frac{n\pi y}{a} + b_n \cosh \frac{n\pi y}{a} \tag{2.25}$$

Imposing the boundary condition in Eq. (2.13c),

$$Y(0) = 0 \qquad \rightarrow \qquad b_n = 0$$

so that

$$Y_n(y) = a_n \sinh \frac{n\pi y}{a} \tag{2.26}$$

Substituting Eqs. (2.23) and (2.26) into Eq. (2.9), we obtain

$$V_n(x, y) = X_n(x)Y_n(y) = a_n \sin \frac{n\pi x}{a} \sinh \frac{n\pi y}{a} \,, \tag{2.27}$$

which satisfies Eq. (2.7) and the three homogeneous boundary conditions in Eqs. (2.8a), (2.8b), and (2.8c). By the superposition principle, a linear combination of the solutions V_n, each with different values of n and arbitrary coefficient a_n, is also a solution of Eq. (2.7). Thus we may represent the solution V of Eq. (2.7) as an infinite series in the function V_n, i,e.,

$$V(x, y) = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi x}{a} \sinh \frac{n\pi y}{a} \tag{2.28}$$

We now determine the coefficient a_n by imposing the inhomogeneous boundary condition in Eq. (2.8d) on Eq. (2.28). We get

$$V(x, b) = V_o = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi x}{a} \sinh \frac{n\pi b}{a} \,, \tag{2.29}$$

which is Fourier sine expansion of V_o. Hence,

$$a_n \sinh \frac{n\pi b}{a} = \frac{2}{b} \int_0^b V_o \sin \frac{n\pi x}{a} \, dx = \frac{2V_o}{n\pi}(1 - \cos n\pi)$$

or

$$a_n = \begin{cases} \dfrac{4V_o}{n\pi} \dfrac{1}{\sinh \frac{n\pi b}{a}}, & n = \text{odd} \,, \\ 0, & n = \text{even} \end{cases} \tag{2.30}$$

Substitution of Eq. (2.30) into Eq. (2.28) gives the complete solution as

$$V(x, y) = \frac{4V_o}{\pi} \sum_{n=\text{odd}}^{\infty} \frac{\sin\frac{n\pi x}{a} \sinh\frac{n\pi y}{a}}{n \sinh\frac{n\pi b}{a}} \tag{2.31a}$$

By replacing n by $2k - 1$, Eq. (2.31a) may be written as

$$\boxed{V(x, y) = \frac{4V_o}{\pi} \sum_{k=1}^{\infty} \frac{\sin\frac{n\pi x}{a} \sinh\frac{n\pi y}{a}}{n \sinh\frac{n\pi b}{a}}, \qquad n = 2k - 1} \tag{2.31b}$$

2.3.2 Wave Equation

The time dependence has been taken care of in Section 2.2. We are left with solving the Helmholtz equation

$$\nabla^2 U + k^2 U = 0 \tag{2.5b}$$

In rectangular coordinates, Eq. (2.5b) becomes

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} + k^2 U = 0 \tag{2.32}$$

We let

$$U(x, y, z) = X(x)Y(y)Z(z) \tag{2.33}$$

Substituting Eq. (2.33) into Eq. (2.32) and dividing by XYZ, we obtain

$$\frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z} + k^2 = 0 \tag{2.34}$$

Each term must be equal to a constant since each term depends only on the corresponding variable; X on x, etc. We conclude that

$$\frac{X''}{X} = -k_x^2, \qquad \frac{Y''}{Y} = -k_y^2, \qquad \frac{Z''}{Z} = -k_z^2 \tag{2.35}$$

so that Eq. (2.34) reduces to

$$k_x^2 + k_y^2 + k_z^2 = k^2 \tag{2.36}$$

Notice that there are four separation constants k, k_x, k_y, and k_z since we have four variables t, x, y, and z. But from Eq. (2.36), one is related to the other three so that only three separation constants are independent. As mentioned earlier, the number of independent separation constants is generally one less than the number of independent variables involved. The ordinary differential equations in Eq. (2.35) have solutions

$$X = A_1 e^{jk_x x} + A_2 e^{-jk_x x} \tag{2.37a}$$

(2.37b)

$$+ A_4 e^{jk_y y} \tag{2.37c}$$

or

$$Y = B_3 \sin k_y y + B_4 \cos k_y y \ , \tag{2.37d}$$

$$Z = A_5 e^{jk_z z} + A_6 e^{-jk_z z} \tag{2.37e}$$

or

$$Z = B_5 \sin k_z z + B_6 \cos k_z z \ , \tag{2.37f}$$

Various combinations of X, Y, and Z will satisfy Eq. (2.5b). Suppose we choose

$$X = A_1 e^{jk_x x}, \qquad Y = A_3 e^{jk_y y}, \qquad Z = A_5 e^{jk_z z} \ , \tag{2.38}$$

then

$$U(x, y, z) = A e^{j(k_x x + k_y y + k_z z)} \tag{2.39}$$

or

$$U(\mathbf{r}) = A e^{j\mathbf{k}\cdot\mathbf{r}} \tag{2.40}$$

Introducing the time dependence of Eq. (2.6a) gives

$$\boxed{\Phi(x, y, z, t) = A e^{j(\mathbf{k}\cdot\mathbf{r} + \omega t)}} \tag{2.41}$$

where $\omega = kc$ is the angular frequency of the wave and k is given by Eq. (2.36). The solution in Eq. (2.41) represents a plane wave of amplitude A propagating in the direction of the wave vector $\mathbf{k} = k_x \mathbf{a}_x + k_y \mathbf{a}_y + k_z \mathbf{a}_z$ with velocity c.

Example 2.1

In this example, we would like to show that the method of separation of variables is not limited to a problem with only one inhomogeneous boundary condition as presented in Section 2.3.1. We reconsider the problem of Fig. 2.1, but with four inhomogeneous boundary conditions as in Fig. 2.2(a). □

Solution

The problem can be stated as solving Laplace's equation

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0 \tag{2.42}$$

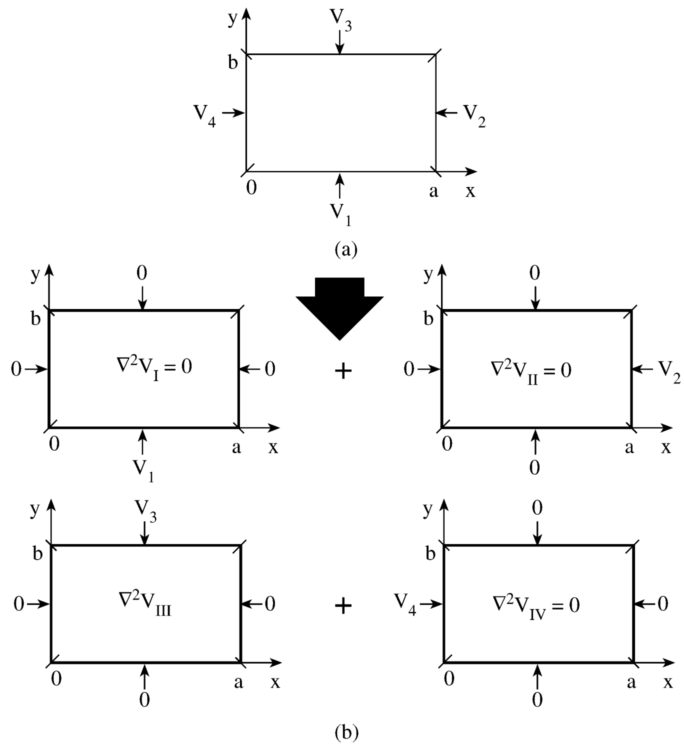


Figure 2.2
Applying the principle of superposition reduces the problem in (a) to those in (b).

subject to the following inhomogeneous Dirichlet conditions:

$$\begin{aligned} V(x,0) &= V_1 \\ V(x,b) &= V_3 \\ V(0,y) &= V_4 \\ V(a,y) &= V_2 \end{aligned} \tag{2.43}$$

Since Laplace's equation is a linear homogeneous equation, the problem can be simplified by applying the superposition principle. If we let

$$V = V_I + V_{II} + V_{III} + V_{IV} \ , \tag{2.44}$$

we may reduce the problem to four simpler problems, each of which is associated with one of the inhomogeneous conditions. The reduced, simpler problems are illustrated in Fig. 2.2 (b) and stated as follows:

$$\frac{\partial^2 V_I}{\partial x^2} + \frac{\partial^2 V_I}{\partial y^2} = 0 \tag{2.45}$$

subject to

$$\begin{aligned} V_I(x, 0) &= V_1 \\ V_I(x, b) &= 0 \\ V_I(0, y) &= 0 \\ V_I(a, y) &= 0\,; \end{aligned} \tag{2.46}$$

$$\frac{\partial^2 V_{II}}{\partial x^2} + \frac{\partial^2 V_{II}}{\partial y^2} = 0 \tag{2.47}$$

subject to

$$\begin{aligned} V_{II}(x, 0) &= 0 \\ V_{II}(x, b) &= 0 \\ V_{II}(0, y) &= 0 \\ V_{II}(a, y) &= V_2\,; \end{aligned} \tag{2.48}$$

$$\frac{\partial^2 V_{III}}{\partial x^2} + \frac{\partial^2 V_{III}}{\partial y^2} = 0 \tag{2.49}$$

subject to

$$\begin{aligned} V_{III}(x, 0) &= 0 \\ V_{III}(x, b) &= V_3 \\ V_{III}(0, y) &= 0 \\ V_{III}(a, y) &= 0\,; \end{aligned} \tag{2.50}$$

and

$$\frac{\partial^2 V_{IV}}{\partial x^2} + \frac{\partial^2 V_{IV}}{\partial y^2} = 0 \tag{2.51}$$

subject to

$$\begin{aligned} V_{IV}(x, 0) &= 0 \\ V_{IV}(x, b) &= 0 \\ V_{IV}(0, y) &= V_4 \\ V_{IV}(a, y) &= 0 \end{aligned} \tag{2.52}$$

It is obvious that the reduced problem in Eqs. (2.49) and (2.50) with solution V_{III} is the same as that in Fig. 2.1. The other three reduced problems are quite similar. Hence the solutions V_I, V_{II}, and V_{IV} can be obtained by taking the same steps as in

Section 2.3.1 or by a proper exchange of variables in Eq. (2.31). Thus

$$V_I = \frac{4V_1}{\pi} \sum_{n=\text{odd}}^{\infty} \frac{\sin \frac{n\pi x}{a} \sinh \frac{n\pi(b-y)}{a}}{n \sinh \frac{n\pi b}{a}} , \tag{2.53}$$

$$V_{II} = \frac{4V_2}{\pi} \sum_{n=\text{odd}}^{\infty} \frac{\sin \frac{n\pi x}{b} \sinh \frac{n\pi y}{b}}{n \sinh \frac{n\pi a}{b}} , \tag{2.54}$$

$$V_{III} = \frac{4V_3}{\pi} \sum_{n=\text{odd}}^{\infty} \frac{\sin \frac{n\pi x}{a} \sinh \frac{n\pi y}{a}}{n \sinh \frac{n\pi b}{a}} , \tag{2.55}$$

$$V_{IV} = \frac{4V_4}{\pi} \sum_{n=\text{odd}}^{\infty} \frac{\sin \frac{n\pi(a-x)}{b} \sinh \frac{n\pi y}{b}}{n \sinh \frac{n\pi a}{b}} \tag{2.56}$$

We obtain the complete solution by substituting Eqs. (2.53) to (2.56) in Eq. (2.44). ■

Example 2.2

Find the product solution of the diffusion equation

$$\Phi_t = k\Phi_{xx}, \qquad 0 < x < 1, \quad t > 0 \tag{2.57}$$

subject to the boundary conditions

$$\Phi(0,t) = 0 = \Phi(1,t), \qquad t > 0 \tag{2.58}$$

and initial condition

$$\Phi(x,0) = 5\sin 2\pi x, \qquad 0 < x < 1 \tag{2.59}$$

Solution

Let

$$\Phi(x,t) = X(x)T(t) \tag{2.60}$$

Substitute this into Eq. (2.57) and divide by kXT to obtain

$$\frac{T'}{kT} = \frac{X''}{X} = \lambda$$

where λ is the separation constant. Thus

$$X'' - \lambda X = 0 \tag{2.61}$$

$$T' - \lambda kT = 0 \tag{2.62}$$

As usual, in order for the solution of Eq. (2.61) to satisfy Eq. (2.58), we must choose $\lambda = -\beta^2 = -n^2\pi^2$ so that $n = 1, 2, 3, \ldots$ and

$$X_n(x) = \sin n\pi x \tag{2.63}$$

Equation (2.62) becomes

$$T' + kn^2\pi^2 T = 0 ,$$

which has solution

$$T_n(t) = e^{-kn^2\pi^2 t} \tag{2.64}$$

Substituting Eqs. (2.63) and (2.64) into Eq. (2.60),

$$\Phi_n(x, t) = a_n \sin n\pi x \exp(-kn^2\pi^2 t)$$

where the coefficients a_n are to be determined from the initial condition in Eq. (2.59). The complete solution is a linear combination of Φ_n, i.e.,

$$\Phi(x, t) = \sum_{n=1}^{\infty} a_n \sin n\pi x \exp(-kn^2\pi^2 t)$$

This satisfies Eq. (2.59) if

$$\Phi(x, 0) = \sum_{n=1}^{\infty} a_n \sin n\pi x = 5 \sin 2\pi x \tag{2.65}$$

The coefficients a_n are determined as $(T = 1)$

$$a_n = \frac{2}{T} \int_0^1 5 \sin 2\pi x \sin n\pi x \, dx = \begin{cases} 5, & n = 2 \\ 0, & n \neq 0 \end{cases}$$

Alternatively, by comparing the middle term in Eq. (2.65) with the last term, the two are equal only when $n = 2$, $a_n = 5$, otherwise $a_n = 0$. Hence the solution of the diffusion problem becomes

$$\Phi(x, t) = 5 \sin 2\pi t \exp(-4k\pi^2 t) \qquad \blacksquare$$

2.4 Separation of Variables in Cylindrical Coordinates

Coordinate geometries other than rectangular Cartesian are used to describe many EM problems whenever it is necessary and convenient. For example, a problem having cylindrical symmetry is best solved in cylindrical system where the coordinate variables (ρ, ϕ, z) are related as shown in Fig. 2.3 and $0 \leq \rho \leq \infty, 0 \leq \phi < 2\pi, -\infty \leq z \leq \infty$. In this system, the wave equation (2.5b) becomes

$$\nabla^2 U + k^2 U = \frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial U}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 U}{\partial \phi^2} + \frac{\partial^2 U}{\partial z^2} + k^2 U = 0 \tag{2.66}$$

As we did in the previous section, we shall first solve Laplace's equation $(k = 0)$ in two dimensions before we solve the wave equation.

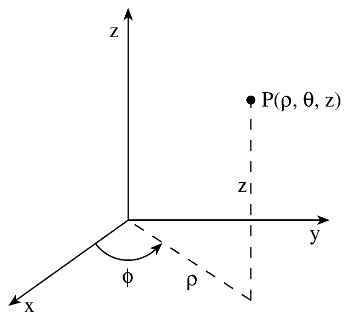


Figure 2.3
Coordinate relations in a cylindrical system.

2.4.1 Laplace's Equation

Consider an infinitely long conducting cylinder of radius a with the cross section shown in Fig. 2.4. Assume that the upper half of the cylinder is maintained at potential V_o while the lower half is maintained at potential $-V_o$. This is a Laplacian problem in two dimensions. Hence we need to solve for $V(\rho, \phi)$ in Laplace's equation

$$\nabla^2 V = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial V}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 V}{\partial \phi^2} = 0 \tag{2.67}$$

subject to the inhomogeneous Dirichlet boundary condition

$$V(a, \phi) = \begin{cases} V_o, & 0 < \phi < \pi \\ -V_o, & \pi < \phi < 2\pi \end{cases} \tag{2.68}$$

We let

$$V(\rho, \phi) = R(\rho)F(\phi) \tag{2.69}$$

Substituting Eq. (2.69) into Eq. (2.67) and dividing through by RF/ρ^2 result in

$$\frac{\rho}{R}\frac{d}{d\rho}\left(\rho \frac{dR}{d\rho}\right) + \frac{1}{F}\frac{d^2 F}{d\phi^2} = 0$$

or

$$\frac{\rho^2}{R}\frac{d^2 R}{d\rho^2} + \frac{\rho}{R}\frac{dR}{d\rho} = -\frac{1}{F}\frac{d^2 F}{d\phi^2} = \lambda^2 \tag{2.70}$$

where λ is the separation constant. Thus the separated equations are:

$$F'' + \lambda^2 F = 0 \tag{2.71a}$$

$$\rho^2 R'' + \rho R' - \lambda^2 R = 0 \tag{2.71b}$$

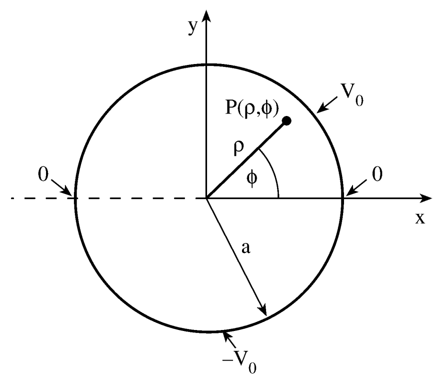


Figure 2.4
A two-dimensional Laplacian problem in cylindrical coordinates.

It is evident that Eq. (2.71a) has the general solution of the form

$$F(\phi) = c_1 \cos(\lambda\phi) + c_2 \sin(\lambda\phi) \tag{2.72}$$

From the boundary conditions of Eq. (2.68), we observe that $F(\phi)$ must be a periodic, odd function. Thus $c_1 = 0$, $\lambda = n$, a real integer, and hence Eq. (2.72) becomes

$$F_n(\phi) = c_2 \sin n\phi \tag{2.73}$$

Equation (2.71b), known as the *Cauchy-Euler equation,* can be solved by making a substitution $\rho = e^u$ and reducing it to an equation with constant coefficients. This leads to

$$R_n(\rho) = c_3\rho^n + c_4\rho^{-n}, \qquad n = 1, 2, \dots \tag{2.74}$$

Note that case $n = 0$ is excluded; if $n = 0$, we obtain $R(\rho) = \ln \rho +$ constant, which is not finite at $\rho = 0$. For the problem of a coaxial cable, $a < \rho < b$, $\rho \neq 0$ so that case $n = 0$ is the only solution. However, for the problem at hand, $n = 0$ is not acceptable.

Substitution of Eqs. (2.73) and (2.74) into Eq. (2.69) yields

$$V_n(\rho, \phi) = \sin n\phi (A_n\rho^n + B_n\rho^{-n}) \tag{2.75}$$

where A_n and B_n are constants to be determined. As usual, it is possible by the superposition principle to form a complete series solution

$$V(\rho, \phi) = \sum_{n=1}^{\infty} (A_n\rho^n + B_n\rho^{-n}) \sin n\phi \tag{2.76}$$

For $\rho < a$, inside the cylinder, V must be finite as $\rho \to 0$ so that $B_n = 0$. At $\rho = a$,

$$V(a, \phi) = \sum_{n=1}^{\infty} A_n a^n \sin n\phi = \begin{cases} V_o, & 0 < \phi < \pi \\ -V_o, & \pi < \phi < 2\pi \end{cases} \tag{2.77}$$

Multiplying both sides by $\sin m\phi$ and integrating over $0 < \phi < 2\pi$, we get

$$\int_0^\pi V_o \sin m\phi \, d\phi - \int_\pi^{2\pi} V_o \sin m\phi \, d\phi = \sum_{n=1}^{\infty} A_n a^n \int_0^{2\pi} \sin n\phi \sin m\phi \, d\phi$$

All terms in the right-hand side vanish except when $m = n$. Hence

$$\frac{2V_o}{n}(1 - \cos n\pi) = A_n a^n \int_0^{2\pi} \sin^2 \phi \, d\phi = \pi A_n a^n$$

or

$$A_n = \begin{cases} \dfrac{4V_o}{n\pi a^n}, & n = \text{odd} \\ 0, & n = \text{even} \end{cases} \tag{2.78}$$

Thus,

$$V(\rho, \phi) = \frac{4V_o}{\pi} \sum_{n=\text{odd}}^{\infty} \frac{\rho^n \sin n\phi}{n a^n}, \quad \rho < a \tag{2.79}$$

For $\rho > a$, outside the cylinder, V must be finite as $\rho \to \infty$ so that $A_n = 0$ in Eq. (2.76) for this case. By imposing the boundary condition in Eq. (2.68) and following the same steps as for case $\rho < a$, we obtain

$$B_n = \begin{cases} \dfrac{4V_o a^n}{n\pi}, & n = \text{odd} \\ 0, & n = \text{even} \end{cases} \tag{2.80}$$

Hence,

$$V(\rho, \phi) = \frac{4V_o}{\pi} \sum_{n=\text{odd}}^{\infty} \frac{a^n \sin n\phi}{n \rho^n}, \quad \rho > a \tag{2.81}$$

2.4.2 Wave Equation

Having taken care of the time-dependence in Section 2.2, we now solve Helmholtz's equation (2.66), i.e.,

$$\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial U}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 U}{\partial \phi^2} + \frac{\partial^2 U}{\partial z^2} + k^2 U = 0 \tag{2.66}$$

Let

$$U(\rho, \phi, z) = R(\rho) F(\phi) Z(z) \tag{2.82}$$

Substituting Eq. (2.82) into Eq. (2.66) and dividing by RFZ/ρ^2 yields

$$\frac{\rho}{R}\frac{d}{d\rho}\left(\rho\frac{dR}{d\rho}\right)+\frac{\rho^2}{Z}\frac{d^2Z}{dz^2}+k^2\rho^2=-\frac{1}{F}\frac{d^2F}{d\phi^2}=n^2$$

where $n = 0, 1, 2, \ldots$ and n^2 is the separation constant. Thus

$$F''+n^2F=0 \tag{2.83}$$

and

$$\frac{\rho}{R}\frac{d}{d\rho}\left(\rho\frac{dR}{d\rho}\right)+\frac{\rho^2}{Z}\frac{d^2Z}{dz^2}+k^2\rho^2=n^2 \tag{2.84}$$

Dividing both sides of Eq. (2.84) by ρ^2 leads to

$$\frac{1}{\rho R}\frac{d}{d\rho}\left(\rho\frac{dR}{d\rho}\right)+\left(k^2-\frac{n^2}{\rho^2}\right)=-\frac{1}{Z}\frac{d^2Z}{dz^2}=\mu^2$$

where μ^2 is another separation constant. Hence

$$-\frac{1}{Z}\frac{d^2Z}{dz^2}=\mu^2 \tag{2.85}$$

and

$$\frac{1}{\rho R}\frac{d}{d\rho}\left(\rho\frac{dR}{d\rho}\right)+\left(k^2-\mu^2-\frac{n^2}{\rho^2}\right)=0 \tag{2.86}$$

If we let

$$\lambda^2=k^2-\mu^2\,, \tag{2.87}$$

the three separated equations (2.83), (2.85), and (2.86) become

$$F''+n^2F=0\,, \tag{2.88}$$

$$Z''+\mu^2Z=0\,, \tag{2.89}$$

$$\rho^2R''+\rho R+\left(\lambda^2\rho^2-n^2\right)R=0 \tag{2.90}$$

The solution to Eq. (2.88) is given by

$$F(\phi)=c_1e^{jn\phi}+c_2e^{-jn\phi} \tag{2.91a}$$

or

$$F(\phi)=c_3\sin n\phi+c_4\cos n\phi \tag{2.91b}$$

Similarly, Eq. (2.89) has the solution

$$Z(z) = c_5 e^{jn\mu} + c_6 e^{-jn\mu} \tag{2.92a}$$

or

$$Z(z) = c_7 \sin n\mu + c_8 \cos n\mu \tag{2.92b}$$

To solve Eq. (2.90), we let $x = \lambda\rho$ and replace R by y; the equation becomes

$$x^2 y'' + xy' + (x^2 - n^2)y = 0 \tag{2.93}$$

This is called *Bessel's equation*. It has a general solution of the form

$$y(x) = b_1 J_n(x) + b_2 Y_n(x) \tag{2.94}$$

where $J_n(x)$ and $Y_n(x)$ are, respectively, *Bessel functions* of the first and second kinds of order n and real argument x. Y_n is also called the *Neumann function*. If x in Eq. (2.93) is imaginary so that we may replace x by jx, the equation becomes

$$x^2 y'' + xy' - (x^2 + n^2)y = 0 \tag{2.95}$$

which is called *modified Bessel's equation*. This equation has a solution of the form

$$y(x) = b_3 I_n(x) + b_4 K_n(x) \tag{2.96}$$

where $I_n(x)$ and $K_n(x)$ are respectively *modified Bessel functions* of the first and second kind of order n. For small values of x, Fig. 2.5 shows the sketch of some typical Bessel functions (or cylindrical functions) $J_n(x)$, $Y_n(x)$, $I_n(x)$, and $K_n(x)$.

To obtain the Bessel functions from Eqs (2.93) and (2.95), the method of Frobenius is applied. A detailed discussion is found in Kersten [4] and Myint-U [5]. For the Bessel function of the first kind,

$$\boxed{y = J_n(x) = \sum_{m=0}^{\infty} \frac{(-1)^m (x/2)^{n+2m}}{m!\Gamma(n+m+1)}} \tag{2.97}$$

where $\Gamma(k+1) = k!$ is the Gamma function. This is the most useful of all Bessel functions. Some of its important properties and identities are listed in Table 2.1. For the modified Bessel function of the second kind

$$I_n(x) = j^{-n} J_n(jx) = \sum_{m=U}^{\infty} \frac{(x/2)^{n+2m}}{m!\Gamma(n+m+1)} \tag{2.98}$$

For the Neumann function, when $n > 0$

$$\begin{aligned} Y_n(x) = \frac{2}{\pi} J_n(x) \ln \frac{\gamma x}{2} - \frac{1}{\pi} \sum_{m=0}^{n-1} \frac{(n-m-1)!(x/2)^{2m-n}}{m!} \\ - \frac{1}{\pi} \sum_{m=0}^{\infty} \frac{(-1)^m (x/2)^{n+2m}}{m!\Gamma(n+m+1)} [p(m) + p(n+m)] \end{aligned} \tag{2.99}$$

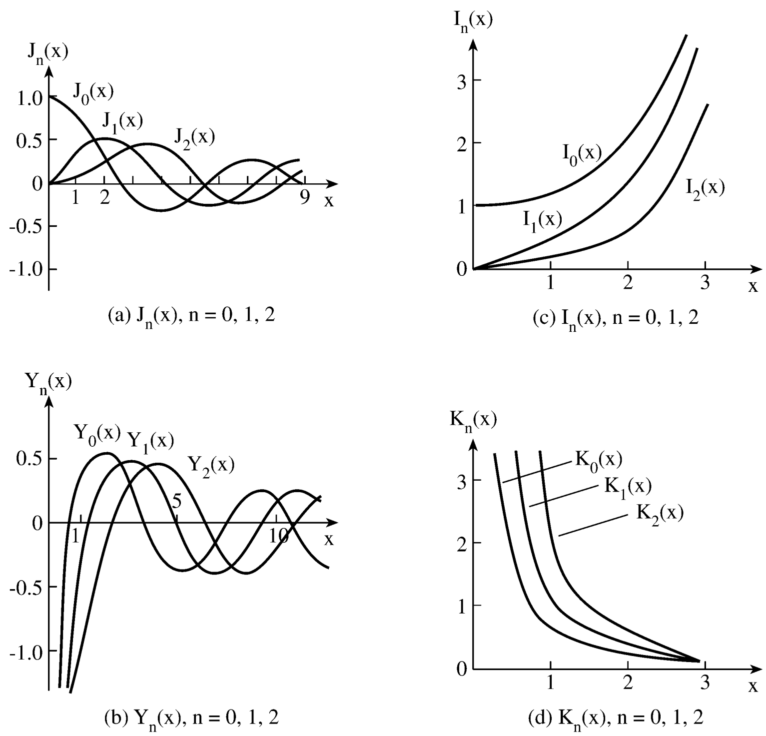


Figure 2.5
Bessel functions.

where $\gamma = 1.781$ is Euler's constant and

$$p(m) = \sum_{k=1}^{m} \frac{1}{k}, \quad p(0) = 0 \tag{2.100}$$

If $n = 0$,

$$Y_0(x) = \frac{2}{\pi} J_0(x) \ln \frac{\gamma x}{2} + \frac{2}{\pi} \sum_{m=0}^{\infty} \frac{(-1)^{m+1}(x/2)^{2m}}{(m!)^2} p(m) \tag{2.101}$$

For the modified Bessel function of the second kind,

$$K_n(x) = \frac{\pi}{2} j^{n+1} \left[J_n(jx) + jY_n(jx) \right] \tag{2.102}$$

Table 2.1 Properties and Identities of Bessel Functions[1] $J_n(x)$

(a) $J_{-n}(x) = (-1)^n J_n(x)$
(b) $J_n(-x) = (-1)^n J_n(x)$
(c) $J_{n+1}(x) = \frac{2n}{x} J_n(x) - J_{n-1}(x)$ (recurrence formula)
(d) $\frac{d}{dx} J_n(x) = \frac{1}{2}[J_{n-1}(x) - J_{n+1}(x)]$
(e) $\frac{d}{dx}[x^n J_n(x)] = x^n J_{n-1}(x)$
(f) $\frac{d}{dx}[x^{-n} J_n(x)] = -x^{-n} J_{n+1}(x)$
(g) $J_n(x) = \frac{1}{\pi}\int_0^\pi \cos(n\theta - x\sin\theta)\, d\theta, \quad n \geq 0$
(h) Fourier-Bessel expansion of $f(x)$: $$f(x) = \sum_{k=1}^{\infty} A_k J_n(\lambda_k x), \quad n \geq 0$$ $$A_k = \frac{2}{[aJ_{n+1}(\lambda_i a)]^2}\int_0^a x f(x) J_n(\lambda_k x)\, dx, \quad 0 < x < a$$ where λ_k are the positive roots in ascending order of magnitude of $J_n(\lambda_i a) = 0$.
(i) $\int_0^a \rho J_n(\lambda_i \rho) J_n(\lambda_j \rho)\, d\rho = \frac{a^2}{2}[J_{n+1}(\lambda_i a)]^2 \delta_{ij}$ where λ_i and λ_j are the positive roots of $J_n(\lambda a) = 0$.

1. Properties (a) to (f) also hold for $Y_n(x)$.

If $n > 0$,

$$K_n(x) = \frac{1}{2}\sum_{m=0}^{n-1} \frac{(-1)^m (n-m-1)!(x/2)^{2m-n}}{m!} + (-1)^n \frac{1}{2}\sum_{m=0}^{\infty} \frac{(x/2)^{n+2m}}{m!(n+m)!}\left[p(m) + p(n+m) - 2\ln\frac{\gamma x}{2}\right] \tag{2.103}$$

and if $n = 0$,

$$K_0(x) = -I_0(x)\ln\frac{\gamma x}{2} + \sum_{m=0}^{\infty} \frac{(x/2)^{2m}}{(m!)^2} p(m) \tag{2.104}$$

Other functions closely related to Bessel functions are *Hankel functions* of the first and second kinds, defined respectively by

$$H_n^{(1)}(x) = J_n(x) + jY_n(x) \tag{2.105a}$$

$$H_n^{(2)}(x) = J_n(x) - jY_n(x) \tag{2.105b}$$

Hankel functions are analogous to functions $\exp(\pm jx)$ just as J_n and Y_n are analogous to cosine and sine functions. This is evident from asymptotic expressions

$$J_n(x) \xrightarrow{x \to \infty} \sqrt{\frac{2}{\pi x}} \cos(x - n\pi/2 - \pi/4) , \tag{2.106a}$$

$$Y_n(x) \xrightarrow{x \to \infty} \sqrt{\frac{2}{\pi x}} \sin(x - n\pi/2 - \pi/4) , \tag{2.106b}$$

$$H_n^{(1)}(x) \xrightarrow{x \to \infty} \sqrt{\frac{2}{\pi x}} \exp[j(x - n\pi/2 - \pi/4)] , \tag{2.106c}$$

$$H_n^{(2)}(x) \xrightarrow{x \to \infty} \sqrt{\frac{2}{\pi x}} \exp[-j(x - n\pi/2 - \pi/4)] , \tag{2.106d}$$

$$I_n(x) \xrightarrow{x \to \infty} \frac{1}{\sqrt{2\pi x}} e^{x} , \tag{2.106e}$$

$$K_n(x) \xrightarrow{x \to \infty} \frac{1}{\sqrt{2\pi x}} e^{-x} \tag{2.106f}$$

With the time factor $e^{j\omega t}$, $H_n^{(1)}(x)$ and $H_n^{(2)}(x)$ represent inward and outward traveling waves, respectively, while $J_n(x)$ or $Y_n(x)$ represents a standing wave. With the time factor $e^{-j\omega t}$, the roles of $H_n^{(1)}(x)$ and $H_n^{(2)}(x)$ are reversed. For further treatment of Bessel and related functions, refer to the works of Watson [6] and Bell [7].

Any of the Bessel functions or related functions can be a solution to Eq. (2.90) depending on the problem. If we choose $R(\rho) = J_n(x) = J_n(\lambda\rho)$ with Eqs. (2.91) and (2.92) and apply the superposition theorem, the solution to Eq. (2.66) is

$$U(\rho, \phi, z) = \sum_n \sum_\mu A_{n\mu} J_n(\lambda\rho) \exp(\pm jn\phi \pm j\mu z) \tag{2.107}$$

Introducing the time dependence of Eq. (2.6a), we finally get

$$\Phi(\rho, \phi, z, t) = \sum_m \sum_n \sum_\mu A_{mn\mu} J_n(\lambda\rho) \exp(\pm jn\phi \pm j\mu z \pm \omega t) , \tag{2.108}$$

where $\omega = kc$.

Example 2.3

Consider the skin effect on a solid cylindrical conductor. The current density distribution within a good conducting wire ($\sigma/\omega\epsilon >> 1$) obeys the diffusion equation

$$\nabla^2 J = \mu\sigma \frac{\partial J}{\partial t}$$

We want to solve this equation for a long conducting wire of radius a. $\square$

Solution

We may derive the diffusion equation directly from Maxwell's equation. We recall that

$$\nabla \times \mathbf{H} = \mathbf{J} + \mathbf{J}_d$$

where $\mathbf{J} = \sigma \mathbf{E}$ is the conduction current density and $\mathbf{J}_d = \frac{\partial \mathbf{D}}{\partial t}$ is the displacement current density. For $\sigma \omega \epsilon >> 1$, $\mathbf{J}_d$ is negligibly small compared with $\mathbf{J}$. Hence

$$\nabla \times \mathbf{H} \simeq \mathbf{J} \tag{2.109}$$

Also,

$$\nabla \times \mathbf{E} = -\mu \frac{\partial \mathbf{H}}{\partial t}$$

$$\nabla \times \nabla \times \mathbf{E} = \nabla \nabla \cdot \mathbf{E} - \nabla^2 \mathbf{E} = -\mu \frac{\partial}{\partial t} \nabla \times \mathbf{H}$$

Since $\nabla \cdot \mathbf{E} = 0$, introducing Eq. (2.109), we obtain

$$\nabla^2 \mathbf{E} = \mu \frac{\partial \mathbf{J}}{\partial t} \tag{2.110}$$

Replacing $\mathbf{E}$ with $\mathbf{J}/\sigma$, Eq. (2.110) becomes

$$\nabla^2 \mathbf{J} = \mu \sigma \frac{\partial \mathbf{J}}{\partial t} \ , \tag{2.111}$$

which is the diffusion equation.

Assuming harmonic field with time factor $e^{j\omega t}$,

$$\nabla^2 \mathbf{J} = j \omega \mu \sigma \mathbf{J} \tag{2.112}$$

For infinitely long wire, Eq. (2.112) reduces to a one-dimensional problem in cylindrical coordinates:

$$\frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial J_z}{\partial \rho} \right) = j \omega \mu \sigma J_z$$

or

$$\rho^2 J_z'' + \rho J_z' - j \omega \mu \sigma \rho^2 J_z = 0 \tag{2.113}$$

Comparing this with Eq. (2.95) shows that Eq. (2.113) is the modified Bessel equation of zero order. Hence the solution to Eq. (2.113) is

$$J_z(\rho) = c_1 I_0(\lambda \rho) + c_2 K_0(\lambda \rho) \tag{2.114}$$

where c_1 and c_2 are constants and

$$\lambda = \sqrt{j \omega \mu \sigma} = j^{1/2} \frac{\sqrt{2}}{\delta} \tag{2.115}$$

and $\delta = \sqrt{\frac{2}{\sigma\mu\omega}}$ is the skin depth. Constant c_2 must vanish if J_z is to be finite at $\rho = 0$. At $\rho = a$,

$$J_z(a) = c_1 I_0(\lambda a) \to c_1 = J_z(a)/I_0(\lambda a)$$

Thus

$$J_z(\rho) = J_z(a)\frac{I_0(\lambda\rho)}{I_0(\lambda a)} \tag{2.116}$$

If we let $\lambda\rho = j^{1/2}\frac{\sqrt{2}}{\delta}\rho = j^{1/2}x$, it is convenient to replace

$$\begin{aligned} I_0(\lambda\rho) &= I_0(j^{1/2}x) = J_0(xe^{j3\pi/4}) \\ &= ber_0(x) + jbei_0(x) \end{aligned} \tag{2.117}$$

where ber_0 and bei_0 are *ber* and *bei* functions of zero order. Ber and ber functions are also known as *Kelvin functions*. For zero order, they are given by

$$ber_0(x) = \sum_{m=0}^{\infty} \frac{\cos(m\pi/2)(x/2)^{2m}}{(m!)^2}\ , \tag{2.118}$$

$$bei_0(x) = \sum_{m=0}^{\infty} \frac{\sin(m\pi/2)(x/2)^{2m}}{(m!)^2} \tag{2.119}$$

Using ber and bei functions, Eq. (2.116) may be written as

$$J_z(\rho) = J_z(a)\frac{ber_0(x) + jbei_0(x)}{ber_0(y) + jbei_0(y)} \tag{2.120}$$

where $x = \sqrt{2}\rho/\delta,\ y = \sqrt{2}a/\delta$. ∎

Example 2.4

A semi-infinitely long cylinder ($z \geq 0$) of radius a has its end at $z = 0$ maintained at $V_o(a^2 - \rho^2),\ 0 \leq \rho \leq a$. Find the potential distribution within the cylinder. □

Solution

The problem is that of finding a function $V(\rho, z)$ satisfying the PDE

$$\nabla^2 V = \frac{\partial^2 V}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial V}{\partial \rho} + \frac{\partial^2 V}{\partial z^2} = 0 \tag{2.121}$$

subject to the boundary conditions:

(i) $V = V_o(a^2 - \rho^2),\ z = 0,\ 0 \leq \rho \leq a,$

(ii) $V \to 0$ as $z \to \infty$, i.e., V is bounded,

(iii) $V = 0$ on $\rho = a$,

(iv) V is finite on $\rho = 0$.

Let $V = R(\rho)Z(z)$ and obtain the separated equations

$$Z'' - \lambda Z = 0 \tag{2.122a}$$

and

$$\rho^2 R'' + \rho R' + \lambda^2 \rho^2 R = 0 \tag{2.122b}$$

where λ is the separated constant. The solution to Eq. (2.122a) is

$$Z_1 = c_1 e^{-\lambda z} + c_2 e^{\lambda z} \tag{2.123}$$

Comparing Eq. (2.122b) with Eq. (2.93) shows that $n = 0$ so that Eq. (2.122b) is Bessel's equation with solution

$$R = c_3 J_0(\lambda\rho) + c_4 Y_0(\lambda\rho) \tag{2.124}$$

Condition (ii) forces $c_2 = 0$, while condition (iv) implies $c_4 = 0$, since $Y_0(\lambda\rho)$ blows up when $\rho = 0$. Hence the solution to Eq. (2.121) is

$$V(\rho, z) = \sum_{n=0}^{\infty} A_n e^{-\lambda_n z} J_0(\lambda_n \rho) \tag{2.125}$$

where A_n and λ_n are constants to be determined using conditions (i) and (iii). Imposing condition (iii) on Eq. (2.125) yields the transcendent equation

$$J_0\left(\lambda_n a\right) = 0 \tag{2.126}$$

Thus λ_n are the positive roots of $J_0(\lambda_n a)$. If we take λ_1 as the first root, λ_2 as the second root, etc., n must start from 1 in Eq. (2.125). Imposing condition (i) on Eq. (2.125), we obtain

$$V(\rho, 0) = V_o\left(a^2 - \rho^2\right) = \sum_{n=1}^{\infty} A_n J_0\left(\lambda_n \rho\right)$$

which is simply the Fourier-Bessel expansion of $V_o(a^2 - \rho^2)$. From Table 2.1, property (h),

$$A_n = \frac{2}{a^2 [J_1(\lambda_n a)]^2} \int_0^a \rho V_o \left(a^2 - \rho^2\right) J_0\left(\lambda_n \rho\right) d\rho \tag{2.127}$$

To evaluate the integral, we utilize property (e) in Table 2.1:

$$\int_0^a x^n J_{n-1}(x)\, dx = x^n J_n(x) \Big|_0^a = a^n J_n(a), \qquad n > 0$$

By changing variables, $x = \lambda\rho$,

$$\int_0^a \rho^n J_{n-1}(\lambda\rho)\, d\rho = \frac{a^n}{\lambda} J_n(\lambda a) \tag{2.128}$$

If $n = 1$,

$$\int_0^a \rho J_0(\lambda\rho)\, d\rho = \frac{a}{\lambda} J_1(\lambda a) \tag{2.129}$$

Similarly, using property (e) in Table 2.1, we may write

$$\int_0^a \rho^3 J_0(\lambda\rho)\, d\rho = \int_0^a \frac{\rho^2}{\lambda} \frac{\partial}{\partial\rho}[\rho J_1(\lambda\rho)]\, d\rho$$

Integrating the right-hand side by parts and applying Eq. (2.128),

$$\begin{aligned}\int_0^a \rho^3 J_0(\lambda\rho)\, d\rho &= \frac{a^3}{\lambda} J_1(\lambda a) - \frac{2}{\lambda}\int_0^a \rho^2 J_1(\lambda\rho)\, d\rho \\ &= \frac{a^3}{\lambda} J_1(\lambda a) - \frac{2a^2}{\lambda^2} J_2(\lambda a)\end{aligned}$$

$J_2(x)$ can be expressed in terms of $J_0(x)$ and $J_1(x)$ using the recurrence relations, i.e., property (c) in Table 2.1:

$$J_2(x) = \frac{2}{x} J_1(x) - J_0(x)$$

Hence

$$\int_0^a \rho^3 J_0(\lambda_n\rho)\, d\rho = \frac{2a^2}{\lambda_n^2}\left[J_0(\lambda_n a) + \left(\frac{a\lambda_n}{2} - \frac{2}{a\lambda_n}\right) J_1(\lambda_n a)\right] \tag{2.130}$$

Substitution of Eqs. (2.129) and (2.130) into Eq. (2.127) gives

$$\begin{aligned}A_n &= \frac{2V_o}{a^2[J_1(\lambda_n a)]^2}\left[\frac{4a}{\lambda_n^3} J_1(\lambda_n a) - \frac{2a^2}{\lambda_n^2} J_0(\lambda_n a)\right] \\ &= \frac{8V_o}{a\lambda_n^3 J_1(\lambda_n a)}\end{aligned}$$

since $J_0(\lambda_n a) = 0$ from Eq. (2.126). Thus the potential distribution is given by

$$V(\rho, z) = \frac{8V_o}{a}\sum_{n=1}^{\infty} \frac{e^{-\lambda_n z} J_0(\lambda_n\rho)}{\lambda_n^3 J_1(\lambda_n a)} \qquad \blacksquare$$

Example 2.5

A plane wave $\mathbf{E} = E_o e^{j(\omega t - kx)}\mathbf{a}_z$ is incident on an infinitely long conducting cylinder of radius a. Determine the scattered field. □

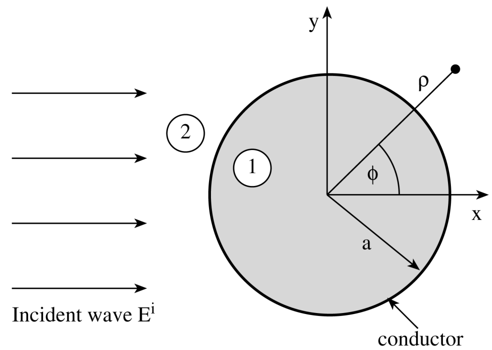


Figure 2.6
Scattering by a conducting cylinder.

Solution
Since the cylinder is infinitely long, the problem is two-dimensional as shown in Fig. 2.6. We shall suppress the time factor $e^{j\omega t}$ throughout the analysis. For the sake of convenience, we need to express the plane wave in terms of cylindrical waves. We let

$$e^{-jx} = e^{-j\rho\cos\phi} = \sum_{n=-\infty}^{\infty} a_n J_n(\rho) e^{jn\phi} \tag{2.131}$$

where a_n are expansion coefficients to be determined. Since $e^{jn\phi}$ are orthogonal functions, multiplying both sides of Eq. (2.131) by $e^{jm\phi}$ and integrating over $0 \le \phi \le 2\pi$ gives

$$\int_0^{2\pi} e^{-j\rho\cos\phi} e^{jm\phi} = 2\pi a_m J_m(\rho)$$

Taking the mth derivative of both sides with respect to ρ and evaluating at $\rho = 0$ leads to

$$2\pi \frac{j^{-m}}{2^m} = 2\pi a_m \frac{1}{2^m} \rightarrow a_m = j^{-m}$$

Substituting this into Eq. (2.131), we obtain

$$e^{-jx} = \sum_{n=-\infty}^{\infty} j^{-n} J_n(\rho) e^{jn\phi}$$

(An alternative, easier way of obtaining this is using the generating function for $J_n(x)$ in Table 2.7.) Thus the incident wave may be written as

$$E_z^i = E_o e^{-jkx} = E_o \sum_{n=-\infty}^{\infty} (-j)^n J_n(k\rho) e^{jn\phi} \tag{2.132}$$

Since the scattered field E_z^s must consist of outgoing waves that vanish at infinity, it contains

$$J_n(k\rho) - jY_n(k\rho) = H_n^{(2)}(k\rho)$$

Hence

$$E_z^s = \sum_{n=-\infty}^{\infty} A_n H_n^{(2)}(k\rho) e^{jn\phi} \tag{2.133}$$

The total field in medium 2 is

$$E_2 = E_z^i + E_z^s$$

while the total field in medium 1 is $E_1 = 0$ since medium 1 is conducting. At $\rho = a$, the boundary condition requires that the tangential components of E_1 and E_2 be equal. Hence

$$E_z^i(\rho = a) + E_z^s(\rho = a) = 0 \tag{2.134}$$

Substituting Eqs. (2.132) and (2.133) into Eq. (2.134),

$$\sum_{n=-\infty}^{\infty} \left[E_o(-j)^n J_n(ka) + A_n H_n^{(2)}(ka) \right] e^{jn\phi} = 0$$

From this, we obtain

$$A_n = -\frac{E_o(-j)^n J_n(ka)}{H_n^{(2)}(ka)}$$

Finally, substituting A_n into Eq. (2.133) and introducing the time factor leads to the scattered wave as

$$\mathbf{E}_z^s = -E_o e^{j\omega t} \mathbf{a}_z \sum_{n=-\infty}^{\infty} (-j)^n \frac{J_n(ka) H_n^{(2)}(k\rho) e^{jn\phi}}{H_n^{(2)}(ka)} \qquad \blacksquare$$

2.5 Separation of Variables in Spherical Coordinates

Spherical coordinates (r, θ, ϕ) may be defined as in Fig. 2.7, where $0 \leq r \leq \infty$, $0 \leq \theta \leq \pi$, $0 \leq \phi < 2\pi$. In this system, the wave equation (2.5b) becomes

$$\nabla^2 U + k^2 U = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial U}{\partial r} \right) + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial \theta} \left(\sin\theta \frac{\partial U}{\partial \theta} \right) + \frac{1}{r^2 \sin^2\theta} \frac{\partial^2 U}{\partial \phi^2} + k^2 U = 0 \tag{2.135}$$

As usual, we shall first solve Laplace's equation in two dimensions and later solve the wave equation in three dimensions.

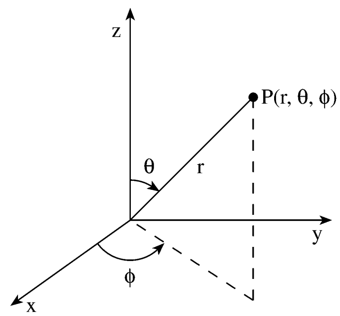


Figure 2.7
Coordinate relation in a spherical system.

2.5.1 Laplace's Equation

Consider the problem of finding the potential distribution due to an uncharged conducting sphere of radius a located in an external uniform electric field as in Fig. 2.8.

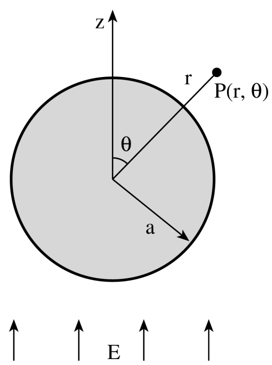


Figure 2.8
An uncharged conducting sphere in a uniform external electric field.

The external electric field can be described as

$$\mathbf{E} = E_o \mathbf{a}_z \tag{2.136}$$

while the corresponding electric potential can be described as

$$V = -\int \mathbf{E} \cdot d\mathbf{l} = -E_o z$$

or

$$V = -E_o r \cos\theta \tag{2.137}$$

where $V(\theta = \pi/2) = 0$ has been assumed. From Eq. (2.137), it is evident that V is independent of ϕ, and hence our problem is solving Laplace's equation in two dimensions, namely,

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right) = 0 \tag{2.138}$$

subject to the conditions

$$V(r,\theta) = -E_o r\cos\theta \quad \text{as} \quad r \to \infty\,, \tag{2.139a}$$

$$V(a,\theta) = 0 \tag{2.139b}$$

We let

$$V(r,\theta) = R(r)H(\theta) \tag{2.140}$$

so that Eq. (2.138) becomes

$$\frac{1}{R}\frac{d}{dr}(r^2R') = -\frac{1}{H\sin\theta}\frac{d}{d\theta}(\sin\theta H') = \lambda \tag{2.141}$$

where λ is the separation constant. Thus the separated equations are

$$r^2R'' + 2rR' - \lambda R = 0 \tag{2.142}$$

and

$$\frac{d}{d\theta}(\sin\theta H') + \lambda\sin\theta H = 0 \tag{2.143}$$

Equation (2.142) is the *Cauchy-Euler equation.* It can be solved by making the substitution $R = r^k$. This leads to the solution

$$R_n(r) = A_n r^n + B_n r^{-(n+1)}, \quad n = 0, 1, 2, \ldots \tag{2.144}$$

with $\lambda = n(n+1)$. To solve Eq. (2.143), we may replace H by y and $\cos\theta$ by x so that

$$\begin{aligned}
\frac{d}{d\theta} = \frac{dx}{d\theta}\frac{d}{dx} &= -\sin\theta\frac{d}{dx} \\
\frac{d}{d\theta}\left(\sin\theta\frac{dH}{d\theta}\right) &= -\sin\theta\frac{d}{dx}\left(\sin\theta\frac{dx}{d\theta}\frac{dH}{dx}\right) \\
&= \sin\theta\frac{d}{dx}\left(\sin^2\theta\frac{dy}{dx}\right) \\
&= \sqrt{1-x^2}\frac{d}{dx}\left[(1-x^2)\frac{dy}{dx}\right]
\end{aligned}$$

Making these substitutions in Eq. (2.143) yields

$$\frac{d}{dx}\left[(1-x^2)\frac{dy}{dx}\right] + n(n+1)y = 0$$

or

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0 \tag{2.145}$$

which is the *Legendre differential equation*. Its solution is obtained by the method of Frobenius [5] as

$$y = c_n P_n(x) + d_n Q_n(x) \tag{2.146}$$

where $P_n(x)$ and $Q_n(x)$ are Legendre functions of the first and second kind, respectively.

$$\boxed{P_n(x) = \sum_{k=0}^{N} \frac{(-1)^k (2n-2k)! x^{n-2k}}{2^n k!(n-k)!(n-2k)!}} \tag{2.147}$$

where $N = n/2$ if n is even and $N = (n-1)/2$ if n is odd. For example,

$$\begin{aligned}
P_0(x) &= 1 \\
P_1(x) &= x = \cos\theta \\
P_2(x) &= \frac{1}{2}(3x^2-1) = \frac{1}{4}(3\cos 2\theta + 1) \\
P_3(x) &= \frac{1}{2}(5x^3-3x) = \frac{1}{8}(5\cos 3\theta + 3\cos\theta) \\
P_4(x) &= \frac{1}{8}(35x^4-30x^2+3) = \frac{1}{64}(35\cos 4\theta + 20\cos 2\theta + 9) \\
P_5(x) &= \frac{1}{8}(63x^5-70x^3+15x) = \frac{1}{128}(30\cos\theta + 35\cos 3\theta + 63\cos 5\theta)
\end{aligned}$$

Some useful identities and properties [5] of Legendre functions are listed in Table 2.2. The Legendre functions of the second kind are given by

$$\begin{aligned}
Q_n(x) = {} & P_n(x)\left[\frac{1}{2}\ln\frac{1+x}{1-x} - p(n)\right] \\
& + \sum_{k=1}^{n} \frac{(-1)^k (n+k)!}{(k!)^2 (n-k)!} p(k) \left[\frac{1-x}{2}\right]^k
\end{aligned} \tag{2.148}$$

where $p(k)$ is as defined in Eq. (2.100). Typical graphs of $P_n(x)$ and $Q_n(x)$ are shown in Fig. 2.9. Q_n are not as useful as P_n since they are singular at $x = \pm 1$ (or $\theta = 0, \pi$) due to the logarithmic term in Eq. (2.148). We use Q_n only when $x \neq \pm 1$ (or $\theta \neq 0, \pi$), e.g., in problems having conical boundaries that exclude the axis from the solution region. For the problem at hand, $\theta = 0, \pi$ is included so that the solution to Eq. (2.143) is

$$H_n(\theta) = P_n(\cos\theta) \tag{2.149}$$

Table 2.2 Properties and Identities of Legendre Functions[1]

(a) For $n \geq 1,\ P_n(1) = 1,\ P_n(-1) = (-1)^n,$

$P_{2n+1} = 0,\ P_{2n}(0) = (-1)^n \frac{(2n)!}{2^{2n}(n!)^2}$

(b) $P_n(-x) = (-1)^n P_n(x)$

(c) $P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n, \qquad n \geq 0$

(Rodriguez formula)

(d) $(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x), \qquad n \geq 1$

(recurrence relation)

(e) $P_n'(x) = xP_{n-1}'(x) + nP_{n-1}(x), \qquad n \geq 1$

(f) $P_n(x) = xP_{n-1}(x) + \frac{x^2-1}{n}P_{n-1}'(x), \qquad n \geq 1$

(g) $P_{n+1}'(x) - P_{n-1}'(x) = (2n+1)P_n(x), \qquad n \geq 1$

or $\quad \int Pn(x)\,dx = \frac{P_{n+1}-P_{n-1}}{2n+1}$

(h) Legendre series expansion of $f(x)$:

$$f(x) = \sum_{n=0}^{\infty} A_n P_n(x), \qquad -1 \leq x \leq 1$$

where

$$A_n = \frac{2n+1}{2}\int_{-1}^{1} f(x)P_n(x)\,dx, \qquad n \geq 0$$

If $f(x)$ is odd,

$$A_n = (2n+1)\int_0^1 f(x)Pn(x)\,dx, \qquad n = 0, 2, 4 \ldots$$

and if $f(x)$ is even,

$$A_n = (2n+1)\int_0^1 f(x)Pn(x)\,dx, \qquad n = 1, 3, 5 \ldots$$

(i) Orthogonality property

$$\int_{-1}^{1} Pn(x)P_m(x)\,dx = \begin{cases} 0, & n \neq m \\ \frac{2}{2n+1}, & n = m \end{cases}$$

1. Properties (d) to (g) are also valid for $Q_n(x)$.

Substituting Eqs. (2.144) and (2.149) into Eq. (2.140) gives

$$V_n(r,\theta) = \left[A_n r^n + B_n r^{-(n+1)}\right] P_n(\cos\theta) \tag{2.150}$$

To determine A_n and B_n we apply the boundary conditions in Eq. (2.139). Since as

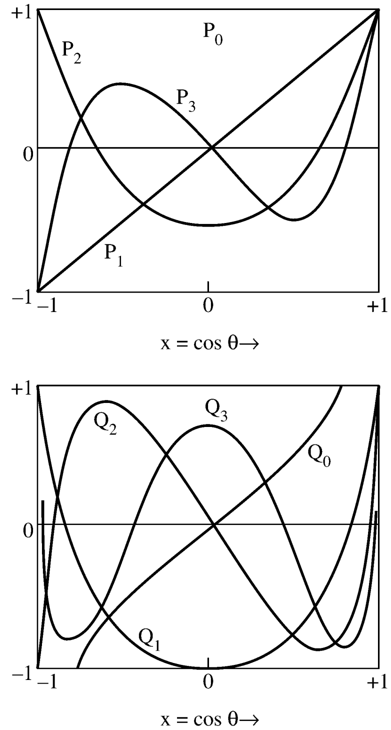


Figure 2.9
Typical Legendre functions of the first and second kinds.

$r \to \infty,\ V = -E_o r \cos\theta$, it follows that $n = 1$ and $A_1 = -E_o$, i.e.,

$$V(r,\theta) = \left(-E_o r + \frac{B_1}{r^2}\right)\cos\theta$$

Also since $V = 0$ when $r = a,\ B_1 = E_o a^3$. Hence the complete solution is

$$V(r,\theta) = -E_o\left(r - \frac{a^3}{r^2}\right)\cos\theta \tag{2.151}$$

The electric field intensity is given by

$$\begin{aligned}\mathbf{E} = -\nabla V &= -\frac{\partial V}{\partial r}\mathbf{a}_r - \frac{1}{r}\frac{\partial V}{\partial \theta}\mathbf{a}_\theta \\ &= E_o\left[1 + \frac{2a^3}{r^3}\right]\cos\theta\,\mathbf{a}_r + E_o\left[1 - \frac{a^3}{r^3}\right]\sin\theta\,\mathbf{a}_\theta\end{aligned} \tag{2.152}$$

2.5.2 Wave Equation

To solve the wave equation (2.135), we substitute

$$U(r, \theta, \phi) = R(r)H(\theta)F(\phi) \tag{2.153}$$

into the equation. Multiplying the result by $r^2 \sin^2\theta / RHF$ gives

$$\frac{\sin^2\theta}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \frac{\sin\theta}{H}\frac{d}{d\theta}\left(\sin\theta\frac{dH}{d\theta}\right) + k^2 r^2 \sin^2\theta = -\frac{1}{F}\frac{d^2F}{d\phi^2} \tag{2.154}$$

Since the left-hand side of this equation is independent of ϕ, we let

$$-\frac{1}{F}\frac{d^2F}{d\phi^2} = m^2, \qquad m = 0, 1, 2, \ldots$$

where m, the first separation constant, is chosen to be nonnegative integer such that U is periodic in ϕ. This requirement is necessary for physical reasons that will be evident later. Thus Eq. (2.154) reduces to

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + k^2r^2 = -\frac{1}{H\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dH}{d\theta}\right) + \frac{m^2}{\sin^2\theta} = \lambda$$

where λ is the second separation constant. As in Eqs. (2.141) to (2.144), $\lambda = n(n+1)$ so that the separated equations are now

$$F'' + m^2 F = 0, \tag{2.155}$$

$$R'' + \frac{2}{r}R' + \left[k^2 - \frac{n(n+1)}{r^2}\right]R = 0\,, \tag{2.156}$$

and

$$\frac{1}{\sin\theta}\frac{d}{d\theta}(\sin\theta H') + \left[n(n+1) - \frac{m^2}{\sin^2\theta}\right]H = 0 \tag{2.157}$$

As usual, the solution to Eq. (2.155) is

$$F(\phi) = c_1 e^{jm\phi} + c_2 e^{-jm\phi} \tag{2.158a}$$

or

$$F(\phi) = c_3 \sin m\phi + c_4 \cos m\phi \tag{2.158b}$$

If we let $R(r) = r^{-1/2}\widetilde{R}(r)$, Eq. (2.156) becomes

$$\widetilde{R}'' + \frac{1}{r}\widetilde{R}' + \left[k^2 - \frac{(n+1/2)^2}{r^2}\right]\widetilde{R} = 0\,,$$

which has the solution

$$\widetilde{R} = Ar^{1/2} z_n(kr) = B Z_{n+1/2}(kr) \tag{2.159}$$

Functions $z_n(x)$ are *spherical Bessel functions* and are related to ordinary Bessel functions $Z_{n+1/2}$ according to

$$z_n(x) = \sqrt{\frac{\pi}{2x}} Z_{n+1/2}(x) \tag{2.160}$$

In Eq. (2.160), $Z_{n+1/2}(x)$ may be any of the ordinary Bessel functions of half-integer order, $J_{n+1/2}(x)$, $Y_{n+1/2}(x)$, $I_{n+1/2}(x)$, $K_{n+1/2}(x)$, $H^{(1)}_{n+1/2}(x)$, and $H^{(2)}_{n+1/2}(x)$, while $z_n(x)$ may be any of the corresponding spherical Bessel functions $j_n(x)$, $y_n(x)$, $i_n(x)$, $k_n(x)$, $h_n^{(1)}(x)$, and $h_n^{(2)}(x)$. Bessel functions of fractional order are, in general, given by

$$\boxed{\begin{aligned}
J_\nu(x) &= \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+\nu}}{2^{2k+\nu} k! \Gamma(\nu + k + 1)} && (2.161)\\
Y_\nu(x) &= \frac{J_\nu(x)\cos(\nu\pi) - J_{-\nu}}{\sin(\nu\pi)} && (2.162)\\
I_\nu(x) &= (-j)^\nu J_\nu(jx) && (2.163)\\
K_\nu(x) &= \frac{\pi}{2}\left[\frac{I_{-\nu} - I_\nu}{\sin(\nu\pi)}\right] && (2.164)
\end{aligned}}$$

where $J_{-\nu}$ and $I_{-\nu}$ are, respectively, obtained from Eqs. (2.161) and (2.163) by replacing ν with $-\nu$. Although ν in Eqs. (2.161) to (2.164) can assume any fractional value, in our specific problem, $\nu = n + 1/2$. Since Gamma function of half-integer order is needed in Eq. (2.161), it is necessary to add that

$$\Gamma(n + 1/2) = \begin{cases} \dfrac{(2n)!}{2^{2n} n!}\sqrt{\pi}\ , & n \geq 0 \\ \dfrac{(-1)^n 2^{2n} n!}{(2n)!}\sqrt{\pi}\ , & n < 0 \end{cases} \tag{2.165}$$

Thus the lower order spherical Bessel functions are as follows:

$$\begin{aligned}
j_0(x) &= \frac{\sin x}{x}\ , & y_0(x) &= -\frac{\cos x}{x}\ ,\\
h_0^{(1)}(x) &= \frac{e^{jx}}{jx}\ , & h_0^{(2)}(x) &= \frac{e^{-jx}}{-jx}\ ,\\
i_0(x) &= \frac{\sinh x}{x}\ , & k_0(x) &= \frac{e^{-x}}{x}\ ,\\
j_1(x) &= \frac{\sin x}{x^2} - \frac{\cos x}{x}\ , & y_1(x) &= -\frac{\cos x}{x^2} - \frac{\sin x}{x}\ ,\\
h_1^{(1)} &= -\frac{(x+j)}{x^2} e^{jx}\ , & h_1^{(2)}(x) &= -\frac{(x-j)}{x^2} e^{-jx}
\end{aligned}$$

Other $z_n(x)$ can be obtained from the series expansion in Eqs. (2.161) and (2.162) or the recurrence relations and properties of $z_n(x)$ presented in Table 2.3.

Table 2.3 Properties and Identities of Spherical Bessel Functions

(a) $z_{n+1} = \frac{(2n+1)}{x} z_n(x) - z_{n-1}(x)$	(recurrence relation)
(b) $\frac{d}{dx} z_n(x) = \frac{1}{2n+1}[n z_{n-1} - (n+1) z_{n+1}(x)]$	
(c) $\frac{d}{dx}[x z_n(x)] = -n z_n(x) + x z_{n-1}(x)$	
(d) $\frac{d}{dx}[x^{n+1} z_n(x)] = -x^{n+1} z_{n-1}(x)$	
(e) $\frac{d}{dx}[x^{-n} z_n(x)] = -x^{-n} z_{n+1}(x)$	
(f) $\int x^{n+2} z_n(x)\, dx = x^{n+2} z_{n+1}(x)$	
(g) $\int x^{1-n} z_n(x)\, dx = -x^{1-n} z_{n-1}(x)$	
(h) $\int x^2 [z_n(x)]^2\, dx = \frac{1}{2} x^3 [z_n(x) - z_{n-1}(x) z_{n+1}(x)]$	

By replacing H in Eq. (2.157) with y, $\cos\theta$ by x, and making other substitutions as we did for Eq. (2.143), we obtain

$$\left(1 - x^2\right) y'' - 2xy' + \left[n(n+1) - \frac{m^2}{1 - x^2}\right] y = 0 , \tag{2.166}$$

which is Legendre's associated differential equation. Its general solution is of the form

$$y(x) = a_{mn} P_n^m(x) + d_{mn} Q_n^m(x) \tag{2.167}$$

where $P_n^m(x)$ and $Q_n^m(x)$ are called associated Legendre functions of the first and second kind, respectively. Equation (2.146) is a special case of Eq. (2.167) when $m = 0$. $P_n^m(x)$ and $Q_n^m(x)$ can be obtained from ordinary Legendre functions $P_n(x)$ and $Q_n(x)$ using

$$P_n^m(x) = \left[1 - x^2\right]^{m/2} \frac{d^m}{dx^m} P_n(x) \tag{2.168}$$

and

$$Q_n^m(x) = \left[1 - x^2\right]^{m/2} \frac{d^m}{dx^m} Q_n(x) \tag{2.169}$$

where $-1 < x < 1$. We note that

$$\begin{aligned} P_n^0(x) &= P_n(x) , \\ Q_n^0(x) &= Q_n(x) , \\ P_n^m(x) &= 0 \quad \text{for} \quad m > n \end{aligned} \tag{2.170}$$

Typical associated Legendre functions are:

$$
\begin{aligned}
P_1^1(x) &= (1-x^2)^{1/2} = \sin\theta \ , \\
P_2^1(x) &= 3x(1-x^2)^{1/2} = 3\cos\theta\sin\theta \ , \\
P_2^2(x) &= 3(1-x^2) = 3\sin^2\theta \ , \\
P_3^1(x) &= \frac{3}{2}(1-x^2)^{1/2}(5x-1) = \frac{3}{2}\sin\theta(5\cos\theta - 1) \ ,
\end{aligned}
$$

$$
\begin{aligned}
Q_1^1(x) &= (1-x^2)^{1/2}\left[\frac{1}{2}\ln\frac{1+x}{1-x} + \frac{x}{1-x^2}\right] \ , \\
Q_2^1 &= (1-x^2)^{1/2}\left[\frac{3x}{2}\ln\frac{1+x}{1-x} + \frac{3x^2-2}{1-x^2}\right] \ , \\
Q_2^2 &= (1-x^2)^{1/2}\left[\frac{3}{2}\ln\frac{1+x}{1-x} + \frac{5x^2-3x^2}{[1-x^2]^2}\right]
\end{aligned}
$$

Higher-order associated Legendre functions can be obtained using Eqs. (2.168) and (2.169) along with the properties in Table 2.4. As mentioned earlier, $Q_n^m(x)$ is unbounded at $x = \pm 1$, and hence it is only used when $x = \pm 1$ is excluded. Substituting Eqs. (2.158), (2.159), and (2.167) into Eq. (2.153) and applying superposition theorem, we obtain

$$
U(r,\theta,\phi,t) = \sum_{n=0}^{\infty}\sum_{m=0}^{n}\sum_{\ell=0}^{\infty} A_{mn\ell}\, z_n(k_{m\ell} r) P_n^m(\cos\theta)\exp(\pm jm\phi \pm j\omega t) \tag{2.171}
$$

Note that the products $H(\theta)F(\phi)$ are known as spherical harmonics.

Example 2.6

A thin ring of radius a carries charge of density ρ. Find the potential at: (a) point $P(0,0,z)$ on the axis of the ring, (b) point $P(r,\theta,\phi)$ in space. □

Solution

Consider the thin ring as in Fig. 2.10.

(a) From elementary electrostatics, at $P(0,0,z)$

$$
V = \int \frac{\rho\, dl}{4\pi\epsilon R}
$$

where $dl = a\,d\phi, \quad R = \sqrt{a^2+z^2}$. Hence

$$
V = \int_0^{2\pi} \frac{\rho a\, d\phi}{4\pi\epsilon[a^2+z^2]^{1/2}} = \frac{a\rho}{2\epsilon[a^2+z^2]^{1/2}} \tag{2.172}
$$

Table 2.4 Properties and Identities of Associated Legendre Functions[1]

(a) $P_m(x) = 0, \quad m > n$

(b) $P_n^m(x) = \frac{(2n-1)x P_{n-1}^m(x) - (n+m-1)P_{n-2}^m(x)}{n-m}$

(recurrence relations for fixed m)

(c) $P_n^m(x) = \frac{2(m-1)x}{(1-x^2)^{1/2}} P_n^{m-1}(x) - (n-m+2)(n+m-1)P_n^{m-2}$

(recurrence relations for fixed n)

(d) $P_n^m(x) = \frac{[1-x^2]^{m/2}}{2^n} \sum_{k=0}^{\left[\frac{m-n}{2}\right]} \frac{(-1)^k(2n-2k)!x^{n-2k-m}}{k!(n-k)!(n-2k-m)!}$

where $[t]$ is the bracket or greatest integer function, e.g., $[3.54] = 3$.

(e) $\frac{d}{dx} P_n^m(x) = \frac{(n+m)P_{n-1}^m(x) - nx P_n^m(x)}{1-x^2}$

(f) $\frac{d}{d\theta} P_n^m(x) = \frac{1}{2}[(n-m+1)(n+m)P_n^{m-1}(x) - P_n^{m+1}(x)]$

(g) $\frac{d}{dx} P_n^m(x) = -\frac{mx P_n^m(x)}{1-x^2}$

$$+ \frac{(1-x^2)^{m/2}}{2^n} \sum_{k=0}^{\left[\frac{m-n}{2}\right]} \frac{(-1)^k(2n-2k)!x^{n-2k-m-1}}{k!(n-k)!(n-2k-m)!}$$

(h) $\frac{d}{d\theta} P_n^m(x) = -(1-x^2)^{1/2} \frac{d}{dx} P_n^m(x)$

(i) The series expansion of $f(x)$:

$$f(x) = \sum_{n=0}^{\infty} A_n P_n^m(x) \ ,$$

where $A_n = \frac{(2n+1)(n-m)!}{2(n+m)!} \int_{-1}^{1} f(x) P_n^m(x)\, dx$

(j) $\frac{d^m}{dx^m} P_n(x)\Big|_{x=1} = \frac{(n+m)!}{2^m m!(n-m)!}$

$\frac{d^m}{dx^m} P_n(x)\Big|_{x=-1} = \frac{(-1)^{n+m}(n+m)!}{2^m m!(n-m)!}$

(k) $P_n^{-m}(x) = (-1)^m \frac{(n-m)!}{(n+m)!} P_n^m(x), \quad m = 0, 1, \ldots, n$

(l) $\int_{-1}^{1} P_n^m(x) P_n^m(x)\, dx = \frac{2}{2n+1} \frac{(n-m)!}{(n+m)!} \delta_{nk},$

where δ_{nk} is the kronecker delta defined by $\delta_{nk} = \begin{cases} 0, & n \neq k \\ 1, & n = k \end{cases}$

1. Properties (b) and (c) are also valid for $Q_n^m(x)$.

(b) To find the potential at $P(r, \theta, \phi)$, we may evaluate the integral for the potential as we did in part (a). However, it turns out that the boundary-value solution is simpler. So we solve Laplace's equation $\nabla^2 V = 0$ where $V(0, 0, z)$ must conform with the result in part (a). From Fig. 2.10, it is evident that V is invariant with ϕ. Hence the solution to Laplace's equation is

$$V = \sum_{n=0}^{\infty} \left[A_n r^n + \frac{B_n}{r^{n+1}} \right] \left[A'_n P_n(u) + B'_n Q_n(u) \right]$$

where $u = \cos\theta$. Since Q_n is singular at $\theta = 0, \pi,\ B'_n = 0$. Thus

$$V = \sum_{n=0}^{\infty} \left[C'_n r^n + \frac{D'_n}{r^{n+1}} \right] P_n(u) \tag{2.173}$$

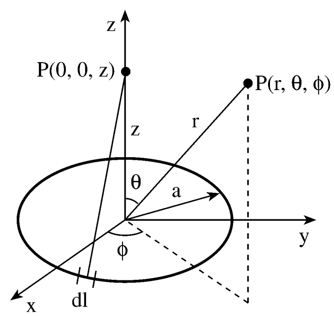


Figure 2.10
Charged ring of Example 2.6.

For $0 \leq r \leq a,\ D'_n = 0$ since V must be finite at $r = 0$.

$$V = \sum_{n=0}^{\infty} C'_n r^n P_n(u) \tag{2.174}$$

To determine the coefficients C'_n, we set $\theta = 0$ and equate V to the result in part (a). But when $\theta = 0,\ u = 1,\ P_n(1) = 1$, and $r = z$. Hence

$$V(0, 0, z) = \frac{a\rho}{2\epsilon[a^2 + z^2]^{1/2}} = \frac{a\rho}{2\epsilon} \sum_{n=0}^{\infty} C_n z^n \tag{2.175}$$

Using the binomial expansion, the term $[a^2 + z^2]^{1/2}$ can be written as

$$\frac{1}{a}\left[1 + \frac{z^2}{a^2}\right]^{-1/2} = \frac{1}{a}\left[1 - \frac{1}{2}(z/a)^2 + \frac{1 \cdot 3}{2 \cdot 4}(z/a)^4 - \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}(z/a)^6 + \dots\right]$$

Comparing this with the last term in Eq. (2.175), we obtain

$$C_0 = 1, \quad C_1 = 0, \quad C_2 = -\frac{1}{2a^2}, \quad C_3 = 0\,,$$
$$C_4 = \frac{1\cdot 3}{2\cdot 4}\frac{1}{a^4}, \quad C_5 = 0, \quad C_6 = -\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\frac{1}{a^6}\,, \ldots$$

or in general,

$$C_{2n} = (-1)^n \frac{(2n)!}{[n!2^n]^2 a^{2n}}$$

Substituting these into Eq. (2.174) gives

$$V = \frac{a\rho}{2\epsilon}\sum_{n=0}^{\infty}\frac{(-1)^n(2n)!}{[n!2^n]^2}(r/a)^{2n}P_{2n}(\cos\theta), \qquad 0 \le r \le a \tag{2.176}$$

For $r \ge a$, $C_n' = 0$ since V must be finite as $r \to \infty$, and

$$V = \sum_{n=0}^{\infty}\frac{D_n'}{r^{n+1}}P_n(u) \tag{2.177}$$

Again, when $\theta = 0$, $u = 1$, $P_n(1) = 1$, $r = z$,

$$V(0,0,z) = \frac{a\rho}{2\epsilon[a^2+z^2]^{1/2}} = \frac{a\rho}{2\epsilon}\sum_{n=0}^{\infty}D_n z^{-(n+1)} \tag{2.178}$$

Using the binomial expansion, the middle term $[a^2+z^2]^{-1/2}$ can be written as

$$\frac{1}{z}\left[1+\frac{a^2}{z^2}\right]^{-1/2} = \frac{1}{z}\left[1-\frac{1}{2}(a/z)^2+\frac{1\cdot 3}{2\cdot 4}(a/z)^4-\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}(a/z)^6+\cdots\right]$$

Comparing this with the last term in Eq. (2.178), we obtain

$$D_0 = 1, \qquad D_1 = 0, \qquad D_2 = -\frac{a^2}{2}, \qquad D_3 = 0\,,$$
$$D_4 = \frac{1\cdot 3}{2\cdot 4}a^4, \qquad D_5 = 0, \qquad D_6 = -\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}a^6, \ldots$$

or in general,

$$D_{2n} = (-1)^n \frac{(2n)!}{[n!2^n]^2}a^{2n}$$

Substituting these into Eq. (2.177) gives

$$V = \frac{a\rho}{2\epsilon r}\sum_{n=0}^{\infty}\frac{(-1)^n(2n)!}{[n!2^n]^2}(a/r)^{2n}P_{2n}(\cos\theta), \qquad r \ge a \tag{2.179}$$

We may combine Eqs. (2.176) and (2.179) to get

$$V = \begin{cases} a \displaystyle\sum_{n=0}^{\infty} g_n (r/a)^{2n} P_{2n}(\cos\theta), & 0 \le r \le a \\ \displaystyle\sum_{n=0}^{\infty} g_n (a/r)^{2n+1} P_{2n}(\cos\theta), & r \ge a \end{cases}$$

where

$$g_n = (-1)^n \frac{\rho}{2\epsilon} \frac{2n!}{[n!2^n]^2} \qquad \blacksquare$$

Example 2.7
A conducting spherical shell of radius a is maintained at potential $V_o \cos 2\phi$; determine the potential at any point inside the sphere. $\square$

Solution
The solution to this problem is somewhat similar to that of the previous problem except that V is a function of ϕ. Hence the solution to Laplace's equation for $0 \le r \le a$ is of the form

$$V = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} (a_{mn} \cos m\phi + b_{mn} \sin m\phi)\,(r/a)^n P_n^m(\cos\theta)$$

Since $\cos m\phi$ and $\sin m\phi$ are orthogonal functions, $a_{mn} = 0 = b_{mn}$ except that $a_{n2} \neq 0$. Hence at $r = a$

$$V_o \cos 2\phi = \cos 2\phi \sum_{n=2}^{\infty} a_{n2} P_n^2(\cos\theta)$$

or

$$V_o = \sum_{n=2}^{\infty} a_{n2} P_n^2(x), \qquad x = \cos\theta$$

which is the Legendre expansion of V_o. Multiplying both sides by $P_m^2(x)$ gives

$$\frac{2}{2n+1} \frac{(n+2)!}{(n-2)!} a_{n2} = V_o \int_{-1}^{1} P_n^2(x)\,dx = V_o \int_{-1}^{1} \left(1 - x^2\right) \frac{d^2}{dx^2} P_n(x)\,dx$$

Integrating by parts twice yields

$$a_{n2} = V_o \frac{2n+1}{2} \frac{(n-2)!}{(n+2)!} \left(2P_n(1) - 2P_n(-1) - 2\int_{-1}^{1} P_n(x)\,dx \right)$$

Using the generating functions for $P_n(x)$ (see Table 2.7 and Example 2.10) it is readily shown that

$$P_n(1) = 1, \qquad P_n(-1) = (-1)^n$$

Also

$$\int_{-1}^{1} P_n(x)\,dx = \int_{-1}^{1} P_0(x) P_n(x)\,dx = 0$$

by the orthogonality property of $P_n(x)$. Hence

$$a_{n2} = V_o(2n+1)\frac{(n-2)!}{(n+2)!}\left[1-(-1)^n\right]$$

and

$$V = V_o \cos 2\phi \sum_{n=2}^{\infty} (2n+1)\frac{(n-2)!}{(n+2)!}\left[1-(-1)^n\right](r/a)^n P_n^2(\cos\theta) \quad \blacksquare$$

Example 2.8

Express: (a) the plane wave e^{jz} and (b) the cylindrical wave $J_0(\rho)$ in terms of spherical wave functions. □

Solution

(a) Since $e^{jz} = e^{jr\cos\theta}$ is independent of ϕ and finite at the origin, we let

$$e^{jz} = e^{jr\cos\theta} = \sum_{n=0}^{\infty} a_n j_n(r) P_n(\cos\theta) \tag{2.180}$$

where a_n are the expansion coefficients. To determine a_n, we multiply both sides of Eq. (2.180) by $P_m(\cos\theta)\sin\theta$ and integrate over $0 < \theta < \pi$:

$$\begin{aligned}\int_0^{\pi} e^{jr\cos\theta} P_m(\cos\theta)\sin\theta d\theta &= \sum_{n=0}^{\infty} a_n j_n(r) \int_{-1}^{1} P_n(x) P_m(x)\,dx \\ &= \begin{cases} 0, & n \neq m \\ \frac{2}{2n+1} a_n j_n(r), & n = m \end{cases}\end{aligned}$$

where the orthogonality property (i) of Table 2.2 has been utilized. Taking the nth derivative of both sides and evaluating at $r = 0$ gives

$$j^n \int_0^{\pi} \cos^n\theta P_n(\cos\theta)\sin\theta d\theta = \frac{2}{2n+1} a_n \frac{d^n}{dr^n} j_n(r)\bigg|_{r=0} \tag{2.181}$$

The left-hand side of Eq. (2.181) yields

$$j^n \int_{-1}^{1} x^n P_n(x)\,dx = \frac{2^{n+1}(n!)^2}{(2n+1)!} j^n \tag{2.182}$$

To evaluate the right-hand side of Eq. (2.181), we recall that

$$j_n(r) = \sqrt{\frac{\pi}{2x}} J_{n+1/2}(r) = \sqrt{\frac{\pi}{2}} \sum_{m=0}^{\infty} \frac{(-1)^m r^{2m+n}}{m!\Gamma(m+n+3/2)2^{2m+n+1/2}}$$

Hence

$$\left.\frac{d^n}{dr^n} j_n(r)\right|_{r=0} = \sqrt{\frac{\pi}{2}} \frac{n!}{\Gamma(n+3/2)2^{n+1/2}} = \frac{2^n (n!)^2}{(2n+1)!} \tag{2.183}$$

Substituting Eqs. (2.182) and (2.183) into Eq. (2.181) gives

$$a_n = j^n(2n+1)$$

Thus

$$e^{jz} = e^{jr\cos\theta} = \sum_{n=0}^{\infty} j^n (2n+1) j_n(r) P_n(\cos\theta) \tag{2.184}$$

(b) Since $J_0(\rho) = J_0(r\sin\theta)$ is even, independent of ϕ, and finite at the origin,

$$J_0(\rho) = J_0(r\sin\theta) = \sum_{n=0}^{\infty} b_n j_{2n}(r) P_{2n}(\cos\theta) \tag{2.185}$$

To determine the coefficients of expansion b_n, we multiply both sides by $P_m(\cos\theta)\sin\theta$ and integrate over $0 < \theta < \pi$. We obtain

$$\int_0^{\pi} J_0(r\sin\theta) P_m(\cos\theta)\sin\theta\, d\theta = \begin{cases} 0, & m \neq 2n \\ \dfrac{2b_n}{4n+1} j_{2n}(r), & m = 2n \end{cases}$$

Differentiating both sides $2n$ times with respect to r and setting $r = 0$ gives

$$b_n = \frac{(-1)^n (4n+1)(2n-1)!}{2^{2n-1} n!(n-1)!}$$

Hence

$$J_0(\rho) = \sum_{n=0}^{\infty} \frac{(-1)^n (4n+1)(2n-1)!}{2^{2n-1} n!(n-1)!} j_{2n}(r) P_{2n}(\cos\theta) \qquad \blacksquare$$

2.6 Some Useful Orthogonal Functions

Orthogonal functions are of great importance in mathematical physics and engineering. A system of real functions $\Phi_n (n = 0, 1, 2, \dots)$ is said to be *orthogonal with weight* $w(x)$ on the interval (a, b) if

$$\int_a^b w(x)\Phi_m(x)\Phi_n(x)\, dx = 0 \tag{2.186}$$

for every $m \neq n$. For example, the system of functions $\cos(nx)$ is orthogonal with weight 1 on the interval $(0, \pi)$ since

$$\int_0^\pi \cos mx \cos nx \, dx = 0 \, , \quad m \neq n$$

Orthogonal functions usually arise in the solution of partial differential equations governing the behavior of certain physical phenomena. These include Bessel, Legendre, Hermite, Laguerre, and Chebyshev functions. In addition to the orthogonality properties in Eq. (2.186), these functions have many other general properties, which will be discussed briefly in this section. They are very useful in series expansion of functions belonging to very general classes, e.g., Fourier-Bessel series, Legendre series, etc. Although Hermite, Laguerre, and Chebyshev functions are of less importance in EM problems than Bessel and Legendre functions, they are sometimes useful and therefore deserve some attention.

An arbitrary function $f(x)$, defined over interval (a, b), can be expressed in terms of any complete, orthogonal set of functions:

$$f(x) = \sum_{n=0}^{\infty} A_n \Phi_n(x) \tag{2.187}$$

where the expansion coefficients are given by

$$A_n = \frac{1}{N_n} \int_a^b w(x) f(x) \Phi_n(x) \, dx \tag{2.188}$$

and the (weighted) norm N_n is defined as

$$N_n = \int_a^b w(x) \Phi_n^2(x) \, dx \tag{2.189}$$

Simple orthogonality results when $w(x) = 1$ in Eqs. (2.186) to (2.189).

Perhaps the best way to briefly describe the orthogonal functions is in table form. This is done in Tables 2.5 to 2.7. The differential equations giving rise to each function are provided in Table 2.5. The orthogonality relations in Table 2.6 are necessary for expanding a given arbitrary function $f(x)$ in terms of the orthogonal functions as in Eqs. (2.187) to (2.189). Most of the properties of the orthogonal functions can be proved using the generating functions of Table 2.7. To the properties in Tables 2.5 to 2.7 we may add the recurrence relations and series expansion formulas for calculating the functions for specific argument x and order n. These have been provided for $J_n(x)$ and $Y_n(x)$ in Table 2.1 and Eqs. (2.97) and (2.99), for $P_n(x)$ and $Q_n(x)$ in Table 2.2 and Eqs. (2.147) and (2.148), for $j_n(x)$ and $y_n(x)$ in Table 2.3 and Eq. (2.160), and for $P_n^m(x)$ and $Q_n^m(x)$ in Table 2.4 and Eqs. (2.168) and (2.169). For Hermite polynomials, the series expansion formula is

$$H_n(x) = \sum_{k=0}^{[n/2]} \frac{(-1)^k n! (2x)^{n-2k}}{k!(n-2k)!} \tag{2.190}$$

Table 2.5 Differential Equations with Solutions

Equations	Solutions	
$x^2y'' + xy' + (x^2 - n^2)y = 0$	$J_n(x)$	Bessel functions of the first kind
	$Y_n(x)$	Bessel functions of the second kind
	$H_n^{(1)}(x)$	Hankel functions of the first kind
	$H_n^{(2)}(x)$	Hankel functions of the second kind
$x^2y'' + xy' - (x^2 + n^2)y = 0$	$I_n(x)$	Modified Bessel functions of the first kind
	$K_n(x)$	Modified Bessel functions of the second kind
$x^2y'' + 2xy' + [x^2 - n(n+1)]y = 0$	$j_n(x)$	Spherical Bessel functions of the first kind
	$y_n(x)$	Spherical Bessel functions of the second kind
$(1-x^2)y'' - 2xy + n(n+1)y = 0$	$P_n(x)$	Legendre polynomials
	$Q_n(x)$	Legendre functions of the second kind
$(1-x^2)y'' - 2xy' + \left[n(n+1) - \frac{m^2}{1-x^2}\right]y = 0$	$P_n^m(x)$	Associated Legendre polynomials
	$Q_n^m(x)$	Associated Legendre functions of the second kind
$y'' - 2xy' + 2ny = 0$	$H_n(x)$	Hermite polynomials
$xy'' + (1-x)y' + ny = 0$	$L_n(x)$	Laguerre polynomials
$xy'' + (m+1-x)y' + ny = 0$	$L_n^m(x)$	Associated Laguerre polynomials
$(1-x^2)y'' - xy' + n^2y = 0$	$T_n(x)$	Chebyshev polynomials of the first kind
	$U_n(x)$	Chebyshev polynomials of the second kind

where $[n/2] = N$ is the largest even integer $\leq n$ or simply the greatest integer function. Thus,

$$H_0(x) = 1, \quad H_1(x) = 2x, \quad H_2(x) = 4x^2 - 2\,, \quad \text{etc.}$$

The recurrence relations are

$$H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x) \tag{2.191a}$$

Table 2.6 Orthogonality Relations

Functions	Relations
Bessel functions	$\int_0^a x J_n(\lambda_i x) J_n(\lambda_j x)\, dx = \frac{a^2}{2}[J_{n+1}(\lambda_i a)]^2 \delta_{ij}$ where λ_i and λ_j are the roots of $J_n(\lambda a) = 0$
Spherical Bessel functions	$\int_{-\infty}^{\infty} j_n(x) j_m(x)\, dx = \frac{\pi}{2n+1}\delta_{mn}$
Legendre polynomials	$\int_{-1}^{1} P_n(x) P_m(x)\, dx = \frac{2}{2n+1}\delta_{mn}$
Associated Legendre polynomials	$\int_{-1}^{1} P_n^k(x) P_m^k(x)\, dx = \frac{2(n+k)!}{(2n+1)(n-k)}\delta_{mn}$
	$\int_{-1}^{1} \frac{P_n^m(x) P_n^k(x)}{1-x^2}\, dx = \frac{(n+m)!}{m(n-m)!}\delta_{mk}$
Hermite polynomials	$\int_{-\infty}^{\infty} e^{-x^2} H_n(x) H_m(x)\, dx = 2^n n!(\sqrt{\pi})\delta_{mn}$
Laguerre polynomials	$\int_0^{\infty} e^{-x} L_n(x) L_m(x)\, dx = \delta_{mn}$
Associated Laguerre polynomials	$\int_0^{\infty} e^{-x} x^k L_n^k(x) L_m^k(x)\, dx = \frac{(n+k)!}{n!}\delta_{mn}$
Chebyshev polynomials	$\int_{-1}^{1} \frac{T_n(x) T_m(x)}{(1-x^2)^{1/2}}\, dx = \begin{cases} 0, & m \neq n \\ \pi/2, & m = n \neq 0 \\ \pi, & m = n = 0 \end{cases}$
	$\int_{-1}^{1} \frac{U_n(x) U_m(x)}{(1-x^2)^{1/2}}\, dx = \begin{cases} 0, & m \neq n \\ \pi/2, & m = n \neq 0 \\ \pi, & m = n = 0 \end{cases}$

and

$$H_n'(x) = 2n H_{n-1}(x) \tag{2.191b}$$

For Laguerre polynomials,

$$L_n(x) = \sum_{k=0}^{n} \frac{n!(-x)^k}{(k!)^2(n-k)!} \tag{2.192}$$

so that

$$L_0(x) = 1, \quad L_1(x) = -x + 1, \quad L_2(x) = \frac{1}{2!}(x^2 - 4x + 2)\,, \quad \text{etc.}$$

The recurrence relations are

$$L_{n+1}(x) = (2n + 1 - x)L_n(x) - n^2 L_{n-1}(x) \tag{2.193a}$$

Table 2.7 Generating Functions

Functions	Generating function $R = [1 - 2xt + t^2]^{1/2}$
Bessel function	$\exp\left[\frac{x}{2}\left(t - \frac{1}{t}\right)\right] = \sum_{n=-\infty}^{\infty} t^n J_n(x)$
Legendre polynomial	$\frac{1}{R} = \sum_{n=0}^{\infty} t^n P_n(x)$
Associated Legendre polynomial	$\frac{(2m)!(1-x^2)^{m/2}}{2^m m! R^{m+1}} = \sum_{n=0}^{\infty} t^n P_{n+m}^m(x)$
Hermite polynomial	$\exp(2tx - t^2) = \sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(x)$
Laguerre polynomial	$\frac{\exp[-xt/(1-t)]}{1-t} = \sum_{n=0}^{\infty} t^n L_n(x)$
Associated Laguerre polynomial	$\frac{\exp[-xt/(1-t)]}{(1-t)^{m+1}} = \sum_{n=0}^{\infty} t^n L_n^m(x)$
Chebyshev polynomial	$\frac{1-t^2}{R^2} = T_0(x) + 2\sum_{n=1}^{\infty} t^n T_n(x)$
	$\frac{\sqrt{1-x^2}}{R^2} = \sum_{n=0}^{\infty} t^n U_{n+1}(x)$

and

$$\frac{d}{dx}L_n(x) = \frac{1}{x}\left[nL_n(x) - n^2 L_{n+1}(x)\right] \tag{2.193b}$$

For the associated Laguerre polynomials,

$$L_n^m(x) = (-1)^m \frac{d^m}{dx^m} L_{n+m}(x) = \sum_{k=0}^{n} \frac{(m+n)!(-x)^k}{k!(n-k)!(m+k)!} \tag{2.194}$$

so that

$$L_1^1(x) = -x + 2, \quad L_2^2(x) = \frac{x^2}{2} - 3x + 3, \quad L_2^2(x) = \frac{x^2}{2} - 4x + 6 \text{ , etc.}$$

Note that $L_n^m(x) = 0, \quad m > n$. The recurrence relations are

$$L_{n+1}^m(x) = \frac{1}{n+1}\left[(2n + m + 1 - x)L_n^m(x) - (n+m)L_{n-1}^m(x)\right] \tag{2.195}$$

For Chebyshev polynomials of the first kind,

$$T_n(x) = \sum_{k=0}^{[n/2]} \frac{(-1)^k n! x^{n-2k}(1-x^2)^k}{(2k)!(n-2k)!}, \quad -1 \le x \le 1 \tag{2.196}$$

so that

$$T_0(x) = 1, \quad T_1(x) = x, \quad T_2(x) = 2x^2 - 1\text{, etc.}$$

The recurrence relation is

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x) \tag{2.197}$$

For Chebyshev polynomials of the second kind,

$$U_n(x) = \sum_{k=0}^{N} \frac{(-1)^{k-1}(n+1)!\, x^{n-2k+2}(1-x^2)^{k-1}}{(2k+1)!(n-2k+2)!}, \qquad -1 \le x \le 1 \tag{2.198}$$

where $N = \left[\frac{n+1}{2}\right]$ so that

$$U_0(x) = 1, \qquad U_1(x) = 2x, \qquad U_2(x) = 4x^2 - 1, \quad \text{etc.}$$

The recurrence relation is the same as that in Eq. (2.197).

For example, if a function $f(x)$ is to be expanded on the interval $(0, \infty)$, Laguerre functions can be used as the orthogonal functions with an exponential weighting function, i.e., $w(x) = e^{-x}$. If $f(x)$ is to be expanded on the interval $(-\infty, \infty)$, we may use Hermite functions with $w(x) = e^{-x^2}$. As we have noticed earlier, if $f(x)$ is defined on the interval $(-1, 1)$, we may choose Legendre functions with $w(x) = 1$. For more detailed treatment of these functions, see Bell [7] or Johnson and Johnson [8].

Example 2.9

Expand the function

$$f(x) = |x|, \qquad -1 \le x \le 1$$

in a series of Chebyshev polynomials. □

Solution

The given function can be written as

$$f(x) = \begin{cases} -x, & -1 \le x < 0 \\ x, & 0 < x \le 1 \end{cases}$$

Let

$$f(x) = \sum_{n=0}^{\infty} A_n T_n(x)$$

where A_n are expansion coefficients to be determined. Since $f(x)$ is an even function, the odd terms in the expansion vanish. Hence

$$f(x) = A_0 + \sum_{n=1}^{\infty} A_{2n} T_{2n}(x)$$

If we multiply both sides by $w(x) = \dfrac{T_{2m}}{\sqrt{1-x^2}}$ and integrate over $-1 \leq x \leq 1$, all terms in the summation vanish except when $m = n$. That is, from Table 2.6, the orthogonality property of $T_n(x)$ requires that

$$\int_{-1}^{1} \frac{T_m(x) T_n(x)}{\left(1-x^2\right)^{1/2}}\, dx = \begin{cases} 0, & m \neq n \\ \pi/2, & m = n \neq 0 \\ \pi, & m = n = 0 \end{cases}$$

Hence

$$\begin{aligned} A_0 &= \frac{1}{\pi} \int_{-1}^{1} \frac{f(x) T_0(x)}{\left(1-x^2\right)^{1/2}}\, dx = \frac{2}{\pi} \int_{0}^{1} \frac{x}{\left(1-x^2\right)^{1/2}}\, dx = \frac{2}{\pi}\,, \\ A_{2n} &= \frac{2}{\pi} \int_{-1}^{1} \frac{f(x) T_{2n}(x)}{\left(1-x^2\right)^{1/2}}\, dx = \frac{4}{\pi} \int_{0}^{1} \frac{x T_{2n}}{\left(1-x^2\right)^{1/2}}\, dx \end{aligned}$$

Since $T_n(x) = \cos(n \cos^{-1} x)$, it is convenient to let $x = \cos\theta$ so that

$$\begin{aligned} A_{2n} &= \frac{4}{\pi} \int_{\pi/2}^{0} \frac{\cos\theta \cos 2n\theta}{\sin\theta} (-\sin\theta\, d\theta) = \frac{4}{\pi} \int_{0}^{\pi/2} \cos\theta \cos 2n\theta\, d\theta \\ &= \frac{4}{\pi} \int_{0}^{\pi/2} \frac{1}{2} [\cos(2n+1)\theta + \cos(2n-1)\theta]\, d\theta = \frac{4}{\pi} \frac{(-1)^{n+1}}{4n^2 - 1} \end{aligned}$$

Hence

$$f(x) = \frac{2}{\pi} + \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{4n^2 - 1} T_{2n}(x) \qquad \blacksquare$$

Example 2.10

Evaluate $\dfrac{P_n^1(x)}{\sin\theta}$ at $x = 1$ and $x = -1$. $\square$

Solution

This example serves to illustrate how the generating functions are useful in deriving some properties of the corresponding orthogonal functions. Since

$$\frac{P_n^1(x)}{\sin\theta} = \frac{P_n^1(x)}{\sqrt{1-x^2}}\,,$$

direct substitution of $x = 1$ or $x = -1$ gives 0/0, which is indeterminate. But $P_n^1(x) = (1-x^2)^{1/2} \frac{d}{dx} P_n$ by definition. Hence

$$\frac{P_n^1(x)}{\sin\theta} = \frac{d}{dx} P_n \ ,$$

i.e., the problem is reduced to evaluating dP_n/dx at $x = \pm 1$. We use the generating function for P_n, namely,

$$\left(1 - 2xt + t^2\right)^{-1/2} = \sum_{n=0}^{\infty} t^n P_n(x)$$

Differentiating both sides with respect to x,

$$\frac{t}{\left(1 - 2xt + t^2\right)^{3/2}} = \sum_{n=0}^{\infty} t^n \frac{d}{dx} P_n \tag{2.199}$$

When $x = 1$,

$$\frac{1}{(1-t)^3} = \sum_{n=0}^{\infty} t^{n-1} \frac{d}{dx} P_n \bigg|_{x=1} \tag{2.200}$$

But

$$(1-t)^{-3} = 1 + 3t + 6t^2 + 10t^3 + 15t^4 + \cdots = \sum_{n=1}^{\infty} \frac{n}{2}(n+1)t^{n-1} \tag{2.201}$$

Comparing this with Eq. (2.200) clearly shows that

$$\frac{d}{dx} P_n \bigg|_{x=1} = n(n+1)/2$$

Similarly, when $x = -1$, Eq. (2.199) becomes

$$\frac{1}{(1+t)^3} = \sum_{n=0}^{\infty} t^{n-1} \frac{d}{dx} P_n \bigg|_{x=-1} \tag{2.202}$$

But

$$(1+t)^{-3} = 1 - 3t + 6t^2 - 10t^3 + 15t^4 - \cdots = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{n}{2}(n+1)t^{n-1}$$

Hence

$$\frac{d}{dx} P_n \bigg|_{x=-1} = (-1)^{n+1} n(n+1)/2 \qquad \blacksquare$$

Example 2.11

Write a program to generate Hermite functions $H_n(x)$ for any argument x and order n. Use the series expansion and recurrence formulas and compare your results. Take $x = 0.5, 0 \leq n \leq 15$.

Solution

The program is shown in Fig. 2.11. Equation (2.190) is used for the series expansion method, while Eq. (2.191a) with $H_0(x) = 1$ and $H_1(x) = 2x$ is used for the recurrence formula. Note that in the program, we have replaced n by $n - 1$ in Eq. (2.191) so that

$$H_n(x) = 2x H_{n-1}(x) - 2(n-1) H_{n-2}(x)$$

The result of the computation is in Table 2.8. In this case, the two methods give identical results. In general, the series expansion method gives results of greater accuracy since error in one computation is not propagated to the next as is the case when using recurrence relations.

Table 2.8 Results of the Program in Fig. 2.11

Values of $H_n(x)$ for $x = 0.5, \quad 0 \leq n \leq 15$			
N	Series Expansion	Recurrence	Difference
0	1.00	1.00	0.00
1	1.00	1.00	0.00
2	−1.00	−1.00	0.00
3	−5.00	−5.00	0.00
4	1.00	1.00	0.00
5	11.00	1.00	0.00
6	31.00	31.00	0.00
7	−461.00	−461.00	0.00
8	−895.00	−895.00	0.00
9	6181.00	6181.00	0.00
10	22591.00	22591.00	0.00
11	−107029.00	−107029.00	0.00
12	−604031.00	−604031.00	0.00
13	1964665.00	1964665.00	0.00
14	17669472.00	17669472.00	0.00
15	−37341152.00	−37341148.00	−4.00

Generating functions such as this is sometimes needed in numerical computations. This example has served to illustrate how this can be done in two ways. Special techniques may be required for very large or very small values of x or n. ∎

```
         ******************************************************
         THIS PROGRAM GENERATES HERMITE'S FUNCTIONS HN(X) IN
         TWO WAYS USING:    1) SERIES EXPANSION
                            2) RECURRENCE RELATION
C        THE TWO METHODS ARE COMPARED
C        X = ARGUMENT (FIXED IN THIS PROGRAM)
C        N = ORDER OF THE FUNCTION  ( 0 < N < 15 IN THIS PROGRAM)
C        ******************************************************

          DIMENSION    HS(0:50), HR(0:50)

          X = 0.5
          NMAX = 15
          WRITE(6,1)
1         FORMAT(2X,68('-'),/)
          WRITE(6,2)
2         FORMAT(3X,'N',14X,'SERIES HS(N)',7X,'RECURRENCE HR(N)',
     2         7X,'DIFFERENCE',/)
          WRITE(6,1)
          DO 60 N=0,NMAX

C         METHOD 1:  SERIES EXPANSION FORMULA

          SUM = 0.0
          CALL FACTORIAL(N.FN)
          CALL GREATEST(N,I)
          DO 10 K=0,I
          M = N - 2*K
          CALL FACTORIAL(M,FM)
          CALL FACTORIAL(K,FK)
          A = ( ((-1.)**K)*FN*((2.*X)**M) )/( FK*FM )
C         FK*FM MAY BE TOO LARGE IF N IS LARGE
          SUM = SUM + A
10        CONTINUE
          HS(N) = SUM

C         METHOD 2:  RECURRENCE FORMULA

          HR(0) = 1.0
          HR(1) = 2.*X
          IF(N-1) 40,40,20
20        DO 30 I=2,N
          HR(I) = 2.0*X*HR(I-1) - 2.*FLOAT(I-1)*HR(I-2)
30        CONTINUE
40        CONTINUE
          DIFFERENCE = HS(N) - HR(N)
          WRITE(6,50) N,HS(N),HR(N),DIFFERENCE
          PRINT *,N,HS(N),HR(N),DIFFERENCE
50        FORMAT(2X,I2,9X,F12.2,9X,F12.2,9X,F10.2,/)
60        CONTINUE
          WRITE(6,1)
          STOP
          END
```

Figure 2.11

Program for Hermite function $H_n(x)$ *(Continued).*

```
C***************************************************************
C
C       SUBROUTINE FOR CALCULATING  N!
        SUBROUTINE FACTORIAL(N,FN)

        FN = 1.0
        IF(N.EQ.0) GO TO 20
        DO 10 I=1,N
        FN = FN*FLOAT(I)
 10     CONTINUE
 20     RETURN
        END
```

```
C***************************************************************
C       SUBROUTINE FOR CALCULATING THE GREATEST INTEGER FUNCTION
C       M = [X]  WHERE X = N/2 IN THIS PARTICULAR CASE
        SUBROUTINE GREATEST(MAX,M)

        A = MAX/2
        M = IFIX(A)
        IF(M) 10,20,20
 10     M = M - 1
 20     RETURN
        END
```

Figure 2.11

(Cont.) **Program for Hermite function $H_n(x)$.**

2.7 Series Expansion

As we have noticed in earlier sections, partial differential equations can be solved with the aid of infinite series and, more generally, with the aid of series of orthogonal functions. In this section we apply the idea of infinite series expansion to those PDEs in which the independent variables are not separable or, if they are separable, the boundary conditions are not satisfied by the particular solutions. We will illustrate the technique in the following three examples.

2.7.1 Poisson's Equation in a Cube

Consider the problem

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = -f(x, y, z) \tag{2.203}$$

subject to the boundary conditions

$$\begin{aligned} V(0, y, z) = V(a, y, z) = V(x, 0, z) = 0 \\ V(x, b, z) = V(x, y, 0) = V(x, y, c) = 0 \end{aligned} \tag{2.204}$$

where $f(x, y, z)$, the source term, is given. We should note that the independent variables in Eq. (2.203) are not separable. However, in Laplace's equation,

$f(x, y, z) = 0$, and the variables are separable. Although the problem defined by Eqs. (2.203) and (2.204) can be solved in several ways, we stress the use of series expansion in this section.

Let the solution be of the form

$$V(x, y, z) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{p=1}^{\infty} A_{mnp} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \sin\frac{p\pi z}{c} \tag{2.205}$$

where the triple sine series is chosen so that the individual terms and the entire series would satisfy the boundary conditions of Eq. (2.204). However, the individual terms do not satisfy either Poisson's or Laplace's equation. Since the expansion coefficients A_{mnp} are arbitrary, they can be chosen such that Eq. (2.205) satisfies Eq. (2.203). We achieve this by substituting Eq. (2.205) into Eq. (2.203). We obtain

$$\begin{aligned}
&-\sum\sum\sum A_{mnp}(m\pi/a)^2 \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \sin\frac{p\pi z}{c} \\
&-\sum\sum\sum A_{mnp}(n\pi/b)^2 \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \sin\frac{p\pi z}{c} \\
&-\sum\sum\sum A_{mnp}(p\pi/c)^2 \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \sin\frac{p\pi z}{c} = -f(x, y, z)
\end{aligned}$$

Multiplying both sides by $\sin(i\pi x/a)\sin(j\pi y/b)\sin(k\pi z/c)$ and integrating over $0 < x < a, 0 < y < b, 0 < z < c$ gives

$$\begin{aligned}
&\sum\sum\sum A_{mnp}\left[(m\pi/a)^2 + (n\pi/b)^2 + (p\pi/c)^2\right] \cdot \\
&\int_0^a \sin\frac{m\pi x}{a} \sin\frac{i\pi x}{a}\,dx \int_0^b \sin\frac{n\pi y}{b} \sin\frac{j\pi y}{b}\,dy \int_0^c \sin\frac{p\pi z}{c} \sin\frac{k\pi z}{c}\,dz \\
&\quad = \int_0^a\int_0^b\int_0^c f(x, y, z) \sin\frac{i\pi x}{a} \sin\frac{j\pi y}{b} \sin\frac{k\pi z}{c}\,dx\,dy\,dz
\end{aligned}$$

Each of the integrals on the left-hand side vanishes except when $m = i, n = j$, and $p = k$. Hence

$$\begin{aligned}
&A_{mnp}\left[(m\pi/a)^2 + (n\pi/b)^2 + (p\pi/c)^2\right]\frac{a}{2}\cdot\frac{b}{2}\cdot\frac{c}{2} \\
&\quad = \int_0^a\int_0^b\int_0^c f(x, y, z) \sin\frac{i\pi x}{a} \sin\frac{j\pi y}{b} \sin\frac{k\pi z}{c}\,dx\,dy\,dz
\end{aligned}$$

or

$$\begin{aligned}
A_{mnp} &= \frac{8}{abc}\left[(m\pi/a)^2 + (n\pi/b)^2 + (p\pi/c)^2\right]^{-1} \cdot \\
&\int_0^a\int_0^b\int_0^c f(x, y, z) \sin\frac{i\pi x}{a} \sin\frac{j\pi y}{b} \sin\frac{k\pi z}{c}\,dx\,dy\,dz
\end{aligned} \tag{2.206}$$

Thus the series expansion solution to the problem is in Eq. (2.205) with A_{mnp} given by Eq. (2.206).

2.7.2 Poisson's Equation in a Cylinder

The problem to be solved is shown in Fig. 2.12, which illustrates a cylindrical metal tank partially filled with charged liquid [9]. To find the potential distribution V in the tank, we let V_ℓ and V_g be the potential in the liquid and gas portions, respectively, i.e.,

$$V = \begin{cases} V_\ell, & 0 < z < b \qquad \text{(liquid)} \\ V_g, & b < z < b + c \qquad \text{(gas)} \end{cases}$$

Thus we need to solve a two-dimensional problem:

$$\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial V_\ell}{\partial \rho}\right) + \frac{\partial^2 V_\ell}{\partial z^2} = -\frac{\rho_v}{\epsilon}\,, \qquad \text{for liquid space} \tag{2.207a}$$

$$\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial V_g}{\partial \rho}\right) + \frac{\partial^2 V_g}{\partial z^2} = 0\,, \qquad \text{for gas space} \tag{2.207b}$$

subject to

$$\begin{aligned} V &= 0, \rho = a \qquad \text{(at the wall)} \\ V_g &= V_\ell, z = b \qquad \text{(at the gas-liquid interface)} \\ \frac{\partial V_g}{\partial z} &= \epsilon_r \frac{\partial V_\ell}{\partial z}, z = b \qquad \text{(at the gas-liquid interface)} \end{aligned}$$

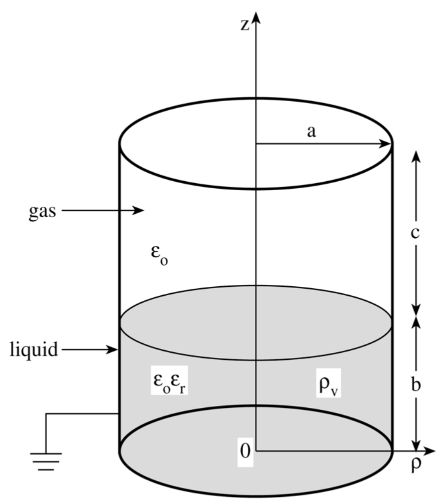


Figure 2.12
A cylindrical metal tank partially filled with charged liquid.

Applying the series expansion techniques, we let

$$V_\ell = \sum_{n=1}^{\infty} J_0(\lambda_n \rho) F_n(z) \tag{2.208a}$$

$$V_g = \sum_{n=1}^{\infty} J_0(\lambda_n \rho) \left[A_n \sinh[\lambda_n(b+c-z)] + B_n \cosh[\lambda_n(b+c-z)]\right] \tag{2.208b}$$

where $F_n(z)$, A_n, and B_n are to be determined.

At $z = b + c$, $V_g = 0$, which implies that $B_n = 0$. Hence, Eq. (2.208b) becomes

$$V_g = \sum_{n=1}^{\infty} A_n J_0(\lambda_n \rho) \sinh[\lambda_n(b+c-z)] \tag{2.209}$$

Substituting Eq. (2.208a) into (2.207b) yields

$$\sum_{n=1}^{\infty} J_0(\lambda_n \rho) \left[F_n'' - \lambda_n^2 F_n\right] = -\frac{\rho_v}{\epsilon}$$

If we let $F_n'' - \lambda_n^2 F_n = G_n$, then

$$\sum_{n=1}^{\infty} G_n J_0(\lambda_n \rho) = -\frac{\rho_v}{\epsilon} \tag{2.210}$$

At $\rho = a$, $V_g = V_\ell = 0$, which makes

$$J_0(\lambda_n a) = 0$$

indicating that λ_n are the roots of J_0 divided by a. Multiplying Eq. (2.210) by $\rho J_0(\lambda_m \rho)$ and integrating over the interval $0 < \rho < a$ gives

$$\sum_{n=1}^{\infty} G_n \int_0^a \rho J_0(\lambda_m \rho) J_0(\lambda_n \rho)\, d\rho = -\frac{\rho_v}{\epsilon} \int_0^a \rho J_0(\lambda_m \rho)\, d\rho$$

The left-hand side is zero except when $m = n$.

$$\int_0^a \rho J_0^2(\lambda_m \rho)\, d\rho = \frac{1}{2} a^2 \left[J_0^2(\lambda_n a) + J_1^2(\lambda_n a)\right] = \frac{a^2}{2} J_1^2(\lambda_n a)$$

since $J_0(\lambda_n a) = 0$. Also,

$$\int_0^a \rho J_0(\lambda_m \rho)\, d\rho = \frac{a}{\lambda_n} J_1(\lambda_n a)$$

Hence

$$G_n \frac{a^2}{2} J_1^2(\lambda_n a) = -\frac{\rho_v}{\epsilon} \frac{a}{\lambda_n} J_1(\lambda_n a)$$

or

$$G_n = -\frac{2\rho_v}{\epsilon a \lambda_n J_1(\lambda_n a)}$$

showing that G_n is a constant. Thus

$$F_n'' - \lambda_n^2 F_n = G_n$$

which is an inhomogeneous ordinary differential equation. Its solution is

$$F_n(z) = C_n \sinh(\lambda_n z) + D_n \cosh(\lambda_n z) - \frac{G_n}{\lambda_n^2}$$

But

$$F_n(0) = 0 \qquad \longrightarrow \qquad D_n = \frac{G_n}{\lambda_n^2}$$

Thus

$$V_\ell = \sum_{n=1}^{\infty} J_0(\lambda_n \rho) \left[C_n \sinh(\lambda_n z) + \frac{G_n}{\lambda_n^2} [\cosh(\lambda_n z) - 1] \right] \tag{2.211}$$

Imposing the conditions at $z = b$, i.e.,

$$V_\ell(\rho, b) = V_g(\rho, b)$$

we obtain

$$A_n \sinh(\lambda_n c) = C_n \sinh(\lambda_n b) + \frac{G_n}{\lambda_n^2} [\cosh(\lambda_n b) - 1] \tag{2.212}$$

Also,

$$\left. \frac{\partial V_g}{\partial z} \right|_{z=b} = \epsilon_r \left. \frac{\partial V_\ell}{\partial z} \right|_{z=b}$$

gives

$$\lambda_n A_n \cosh(\lambda_n c) = -\epsilon_r \lambda_n C_n \cosh(\lambda_n b) - \frac{\epsilon_r G_n}{\lambda_n} \sinh(\lambda_n b) \tag{2.213}$$

Solving Eqs. (2.212) and (2.213), we get

$$\begin{aligned} A_n &= \frac{2\rho_v}{R_n K_n} [\cosh(\lambda_n b) - 1] \\ C_n &= \frac{2\rho_v}{R_n \epsilon_r} [\cosh(\lambda_n b) \cosh(\lambda_n c) + \epsilon_r \sinh(\lambda_n b) \sinh(\lambda_n c) - \cosh(\lambda_n c)] \end{aligned}$$

where

$$\begin{aligned} K_n &= \sinh(\lambda_n b) \cosh(\lambda_n c) + \epsilon_r \cosh(\lambda_n b) \sinh(\lambda_n c) \\ R_n &= \epsilon_o a \lambda_n^3 J_1(\lambda_n a) \end{aligned}$$

Substituting A_n and C_n in Eqs. (2.209) and (2.211), we obtain the complete solution as

$$V_\ell = \sum_{n=1}^{\infty} \frac{2\rho_v}{R_n \epsilon_r} J_0(\lambda_n \rho) \left[\frac{\sinh(\lambda_n z)}{K_n} [\cosh(\lambda_n b) \cosh(\lambda_n c) \right.$$

$$\left. + \epsilon_r \sinh(\lambda_n b) \sinh(\lambda_n c) - \cosh(\lambda_n c)] - \cosh(\lambda_n z) + 1 \right] \tag{2.214a}$$

$$V_g = \sum_{n=1}^{\infty} \frac{2\rho_v}{R_n K_n} J_0(\lambda_n \rho) [\cosh(\lambda_n b) - 1] \sinh[\lambda_n (b + c - z)] \tag{2.214b}$$

2.7.3 Strip Transmission Line

Consider a strip conductor enclosed in a shielded box containing homogeneous medium as shown in Fig. 2.13(a). If TEM mode of propagation is assumed, our problem is reduced to finding V satisfying Laplace's equation $\nabla^2 V = 0$. Due to symmetry, we need only consider one quarter-section of the line as in Fig. 2.13(b). This quadrant can be subdivided into regions 1 and 2, where region 1 is under the center conductor and region 2 is not. We now seek solutions V_1 and V_2 for regions 1 and 2, respectively.

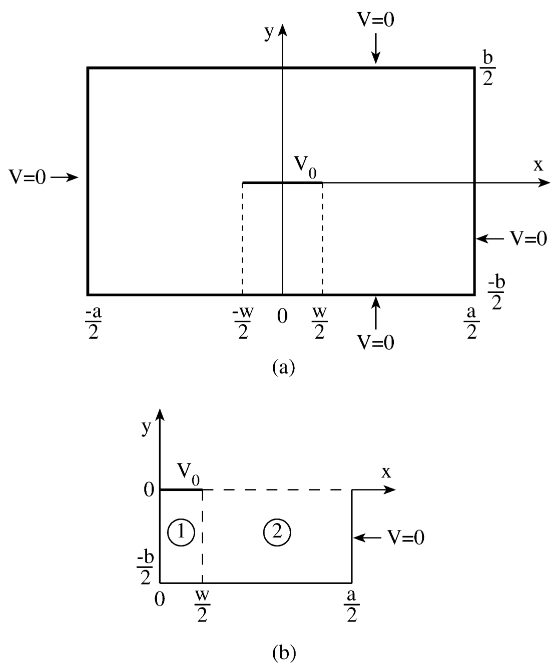


Figure 2.13
Strip line example.

If $w >> b$, region 1 is similar to parallel-plate problem. Thus, we have a one-dimensional problem similar to Eq. (2.14) with solution

$$V_1 = a_1 y + a_2$$

Since $V_1(y = 0) = 0$ and $V_2(y = -b/2) = V_o, a_2 = 0, a_1 = -2V_o/b$. Hence

$$V_1(x, y) = \frac{-2V_o}{b} y \tag{2.215}$$

For region 2, the series expansion solution is of the form

$$V_2(x, y) = \sum_{n=1,3,5}^{\infty} A_n \sin \frac{n\pi y}{b} \sinh \frac{n\pi}{b}(a/2 - x) \ , \tag{2.216}$$

which satisfies Laplace's equation and the boundary condition along the box. Notice that the even-numbered terms could not be included because they do not satisfy the boundary condition requirements about line $y = 0$, i.e., $E_y(y = 0) = -\partial V_2/\partial y\big|_{y=0} \neq 0$. To determine the expansion coefficients A_n in Eq. (2.216), we utilize the fact that V must be continuous at the interface $x = w/2$ between regions 1 and 2, i.e.,

$$V_1(x = w/2, y) = V_2(x = w/2, y)$$

or

$$-\frac{2V_o y}{b} = \sum_{n=\text{odd}}^{\infty} A_n \sin \frac{n\pi y}{b} \sinh \frac{n\pi}{2b}(a - w) \ ,$$

which is Fourier series. Thus,

$$A_n \sinh \frac{n\pi}{2b}(a - w) = -\frac{2}{b} \int_{-b/2}^{b/2} \frac{2V_o y}{b} \sin \frac{n\pi y}{b} \, dy = -\frac{8V_o \sin \frac{n\pi}{2}}{n^2\pi^2}$$

Hence

$$A_n = -\frac{8V_o \sin \frac{n\pi}{2}}{n^2\pi^2 \sinh \frac{n\pi}{2b}(a - w)} \tag{2.217}$$

It is instructive to find the capacitance per unit length C of the strip line using the fact that the energy stored per length is related to C according to

$$W = \frac{1}{2} C V_o^2 \tag{2.218}$$

where

$$W = \frac{1}{2} \int \mathbf{D} \cdot \mathbf{E} \, dv = \frac{1}{2} \epsilon \int |\mathbf{E}|^2 \, dv \tag{2.219}$$

For region 1,

$$\mathbf{E} = -\nabla V = -\frac{\partial V}{\partial x}\mathbf{a}_x - \frac{\partial V}{\partial y}\mathbf{a}_y = \frac{2V_o}{b}\mathbf{a}_y$$

Hence

$$W_1 = \frac{1}{2}\epsilon \int_{x=0}^{w/2} \int_{y=-b/2}^{0} \frac{4V_o^2}{b^2}\, dy\, dx = \frac{\epsilon V_o^2 w}{2b} \tag{2.220}$$

For region 2,

$$\begin{aligned} E_x &= -\frac{\partial V}{\partial x} = \sum \frac{n\pi}{b} A_n \cosh \frac{n\pi}{b}(a/2 - x) \sin \frac{n\pi y}{b} \\ E_y &= -\frac{\partial V}{\partial y} = -\sum \frac{n\pi}{b} A_n \sinh \frac{n\pi}{b}(a/2 - x) \cos \frac{n\pi y}{b} \end{aligned}$$

and

$$\begin{aligned} W_2 &= \frac{1}{2}\epsilon \iint \left(E_x^2 + E_y^2\right) dx\, dy \\ &= \frac{1}{2}\epsilon \int_{y=-b/2}^{0} \int_{x=w/2}^{a/2} \sum_n \sum_m \frac{mn\pi^2}{b^2} A_n A_m \, . \\ &\quad \left[\sinh^2 \frac{m\pi}{b}(a/2 - x) \sinh^2 \frac{n\pi}{b}(a/2 - x) \cos \frac{m\pi y}{b} \cos \frac{n\pi y}{b} \right. \\ &\quad \left. + \cosh^2 \frac{m\pi}{b}(a/2 - x) \cosh^2 \frac{n\pi}{b}(a/2 - x) \sin \frac{m\pi y}{b} \sin \frac{n\pi y}{b}\right] dx\, dy \end{aligned}$$

where the double summation is used to show that we are multiplying two series which may have different indices m and n. Due to the orthogonality properties of sine and cosine functions, all terms vanish except when $m = n$. Thus

$$\begin{aligned} W_2 &= \frac{1}{2}\epsilon \sum_{n=\text{odd}}^{\infty} \frac{n^2\pi^2 A_n^2}{b^2} \cdot \frac{b/2}{2} \int_{w/2}^{a/2} \left[\sinh^2 \frac{n\pi}{b}(a/2 - x) \right. \\ &\quad \left. + \cosh^2 \frac{n\pi}{b}(a/2 - x)\right] dx \\ &= \frac{1}{2}\epsilon \sum_{n=\text{odd}}^{\infty} \frac{n^2\pi^2 A_n^2}{4b} \frac{b}{n\pi} \cosh \frac{n\pi}{2b}(a - w) \sinh \frac{n\pi}{2b}(a - w) \end{aligned}$$

Substituting for A_n gives

$$W_2 = \sum_{n=1,3,5}^{\infty} \frac{8\epsilon V_o^2}{n^3\pi^3} \coth \frac{n\pi}{2b}(a - w) \tag{2.221}$$

The total energy in the four quadrants is

$$W = 4(W_1 + W_2)$$

Thus

$$
\begin{aligned}
C &= \frac{2W}{V_o^2} = \frac{8}{V_o^2}\,(W_1 + W_2) \\
&= \epsilon\left[\frac{4w}{b} + \frac{64}{\pi^3}\sum_{n=1,3,5}^{\infty}\frac{1}{n^3}\coth\frac{n\pi}{2b}(a-w)\right]
\end{aligned}
\tag{2.222}
$$

The characteristic impedance of the lossless line is given by

$$
Z_o = \frac{\sqrt{\mu\epsilon}}{C} = \frac{\sqrt{\mu_r\epsilon_r}}{cC} = \sqrt{\frac{\mu}{\epsilon}}\frac{1}{C/\epsilon}
$$

or

$$
Z_o = \frac{120\pi}{\sqrt{\epsilon_r}\left[\frac{4w}{b} + \frac{64}{\pi^3}\sum_{n=1,3,5}^{\infty}\frac{1}{n^3}\coth\frac{n\pi}{2b}(a-w)\right]}
\tag{2.223}
$$

where $c = 3 \times 10^8$ m/s, the speed of light in vacuum, and $\mu_r = 1$ is assumed.

Example 2.12

Solve the two-dimensional problem

$$
\nabla^2 V = -\frac{\rho_s}{\epsilon_o}
$$

where

$$
\rho_s = x(y-1)\ \text{nC/m}^2
$$

subject to

$$
V(x,0) = 0, \qquad V(x,b) = V_o, \qquad V(0,y) = 0 = V(a,y)
$$

Solution

If we let

$$
\nabla^2 V_1 = 0\,, \tag{2.224a}
$$

subject to

$$
V_1(x,0) = 0, \qquad V_1(x,b) = V_o, \qquad V_1(0,y) = 0 = V(a,y) \tag{2.224b}
$$

and

$$
\nabla^2 V_2 = -\frac{\rho_s}{\epsilon_o}\,, \tag{2.225a}
$$

subject to

$$V_2(x,0)=0, \qquad V_2(x,b)=0, \qquad V_2(0,y)=0=V(a,y) \tag{2.225b}$$

By the superposition principle, the solution to the given problem is

$$V = V_1 + V_2 \tag{2.226}$$

The solution to Eq. (2.224) is already found in Section 2.3.1, i.e.,

$$V_1(x,y) = \frac{4V_o}{\pi} \sum_{n=1,3,5}^{\infty} \frac{\sin\frac{n\pi x}{a}\sinh\frac{n\pi y}{a}}{n\sinh\frac{n\pi b}{a}} \tag{2.227}$$

The solution to Eq. (2.225) is a special case of that of Eq. (2.205). The only difference between this problem and that of Eqs. (2.203) and (2.204) is that this problem is two-dimensional while that of Eqs. (2.203) and (2.204) is three-dimensional. Hence

$$V_2(x,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn} \sin\frac{n\pi x}{a} \sin\frac{n\pi y}{b} \tag{2.228}$$

where, according to Eq. (2.206), A_{mn} is given by

$$\begin{aligned} A_{mn} = {} & \frac{4}{ab}\left[(m\pi/a)^2 + (n\pi/b)^2\right]^{-1} \\ & \cdot \int_0^b \int_0^a f(x,y)\sin\frac{n\pi x}{a}\sin\frac{n\pi y}{b}\,dx\,dy \end{aligned} \tag{2.229}$$

But $f(x,y) = x(y-1)/\epsilon_0$ nC/m^2,

$$\begin{aligned} & \int_0^b \int_0^a f(x,y)\sin\frac{n\pi x}{a}\sin\frac{n\pi y}{a}\,dx\,dy \\ & = \frac{10^{-9}}{\epsilon_o}\int_0^a x\sin\frac{n\pi x}{a}\,dx \int_0^b (y-1)\sin\frac{n\pi y}{b}\,dy \\ & = \frac{10^{-9}}{10^{-9}/36\pi}\left(-\frac{a^2\cos m\pi}{m\pi}\right)\left(-\frac{b^2\cos n\pi}{n\pi} + \frac{b}{n\pi}[\cos n\pi - 1]\right) \\ & = \frac{36\pi(-1)^{m+n}a^2b^2}{mn\pi^2}\left(1 - \frac{1}{b}[1-(-1)^n]\right) \end{aligned} \tag{2.230}$$

since $\cos n\pi = (-1)^n$. Substitution of Eq. (2.230) into Eq. (2.229) leads to

$$A_{mn} = \left[(m\pi/a)^2 + (n\pi/b)^2\right]^{-1} \frac{(-1)^{m+n}144ab}{mn\pi}\left(1 - \frac{1}{b}\left[1-(-1)^n\right]\right) \tag{2.231}$$

Substituting Eqs. (2.227) and (2.228) into Eq. (2.226) gives the complete solution as

$$V(x,y) = \frac{4V_o}{\pi}\sum_{n=1,3,5}^{\infty}\frac{\sin\frac{n\pi x}{a}\sinh\frac{n\pi y}{a}}{n\sinh\frac{n\pi b}{a}} + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn}\sin\frac{n\pi x}{a}\sin\frac{n\pi y}{b} \tag{2.232}$$

where A_{mn} is in Eq. (2.231). ∎

2.8 Practical Applications

The scattering of EM waves by a dielectric sphere, known as the Mie scattering problem due to its first investigator in 1908, is an important problem whose analytic solution is usually referred to in assessing some numerical computations. Though the analysis of the problem is more rigorous, the procedure is similar to that of Example 2.5, where scattering due to a conducting cylinder was treated. Our treatment here will be brief; for an in-depth treatment, consult Stratton [10].

2.8.1 Scattering by Dielectric Sphere

Consider a dielectric sphere illuminated by a plane wave propagating in the z direction and **E** polarized in the x direction as shown in Fig. 2.14. The incident wave

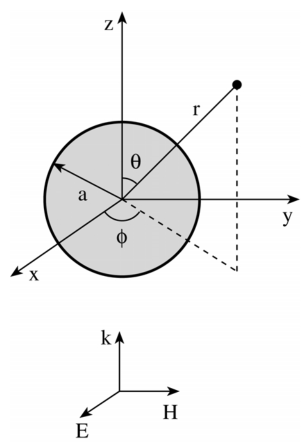


Figure 2.14
Incident EM plane wave on a dielectric sphere.

is described by

$$\mathbf{E}^i = E_o e^{j(\omega t - kz)} \mathbf{a}_x \tag{2.233a}$$

$$\mathbf{H}^i = \frac{E_o}{\eta} e^{j(\omega t - kz)} \mathbf{a}_y \tag{2.233b}$$

The first step is to express this incident wave in terms of spherical wave functions as in Example 2.8. Since

$$\mathbf{a}_x = \sin\theta\cos\phi\mathbf{a}_r + \cos\theta\cos\phi\mathbf{a}_\theta - \sin\phi\mathbf{a}_\phi ,$$

the r-component of $\mathbf{E}^i$, for example, is

$$E_r^i = \cos\phi\sin\theta E_x^i = E_o e^{j\omega t}\frac{\cos\phi}{jkr}\frac{\partial}{\partial\theta}\left(e^{-jkr\cos\theta}\right)$$

Introducing Eq. (2.184),

$$E_r^i = E_o e^{j\omega t} \frac{\cos\phi}{jkr} \sum_{n=0}^{\infty} (-j)^n (2n+1) j_n(kr) \frac{\partial}{\partial\theta} P_n(\cos\theta)$$

But

$$\frac{\partial P_n}{\partial\theta} = P_n^1$$

hence

$$E_r^i = E_o e^{j\omega t} \frac{\cos\phi}{jkr} \sum_{n=1}^{\infty} (-j)^n (2n+1) j_n(kr) P_n^1(\cos\theta) \tag{2.234}$$

where the $n = 0$ term has been dropped since $P_0^1 = 0$. The same steps can be taken to express E_θ^i and E_ϕ^i in terms of the spherical wave functions. The result is

$$\begin{aligned}\mathbf{E}^i &= \mathbf{a}_x E_o e^{j(\omega t - kz)} \\ &= E_o e^{j\omega t} \sum_{n=1}^{\infty} (-j)^n \frac{2n+1}{n(n+1)} \left[\mathbf{M}_n^{(1)}(k) + j\mathbf{N}_n^{(1)}(k)\right]\end{aligned} \tag{2.235a}$$

$$\begin{aligned}\mathbf{H}^i &= \mathbf{a}_y H_o e^{j(\omega t - kz)} \\ &= -\frac{kE_o}{\mu\omega} e^{j\omega t} \sum_{n=1}^{\infty} (-j)^n \frac{2n+1}{n(n+1)} \left[\mathbf{M}_n^{(1)}(k) - j\mathbf{N}_n^{(1)}(k)\right]\end{aligned} \tag{2.235b}$$

where

$$\begin{aligned}\mathbf{M}_n(k) = &\frac{1}{\sin\theta} z_n(kr) P_n^1(\cos\theta) \cos\phi \mathbf{a}_\theta \\ &- z_n(kr) \frac{\partial P_n^1(\cos\theta)}{\partial\theta} \sin\phi \mathbf{a}_\phi\end{aligned} \tag{2.236}$$

$$\begin{aligned}\mathbf{N}_n(k) = &\frac{n(n+1)}{kr} z_n(kr) P_n^1(\cos\theta) \cos\phi \mathbf{a}_r \\ &+ \frac{1}{kr} \frac{\partial}{\partial r}[z_n(kr)] \frac{\partial P_n^1(\cos\theta)}{\partial\theta} \cos\phi \mathbf{a}_\theta \\ &+ \frac{1}{kr\sin\theta} \frac{\partial}{\partial r}[z_n(kr)] P_n^1(\cos\theta) \sin\phi \mathbf{a}_\phi\end{aligned} \tag{2.237}$$

The superscript (1) on the spherical vector functions **M** and **N** in Eq. (2.235) indicates that these functions are constructed with spherical Bessel function of the first kind; i.e., $z_n(kr)$ in Eqs. (2.236) and (2.237) is replaced by $j_n(kr)$ when **M** and **N** are substituted in Eq. (2.235).

The induced secondary field consists of two parts. One part applies to the interior of the sphere and is referred to as the transmitted field, while the other applies to the exterior of the sphere and is called the scattered field. Thus the total field outside the sphere is the sum of the incident and scattered fields. We now construct these fields in a fashion similar to that of the incident field. For the scattered field, we let

$$\mathbf{E}^s = E_o e^{j\omega t} \sum_{n=1}^{\infty} (-j)^n \frac{2n+1}{n(n+1)} \left[a_n \mathbf{M}_n^{(4)}(k) + j b_n \mathbf{N}_n^{(4)}(k) \right] \tag{2.238a}$$

$$\mathbf{H}^s = -\frac{k E_o}{\mu\omega} e^{j\omega t} \sum_{n=1}^{\infty} (-j)^n \frac{2n+1}{n(n+1)} \left[a_n \mathbf{M}_n^{(4)}(k) - j b_n \mathbf{N}_n^{(4)}(k) \right] \tag{2.238b}$$

where a_n and b_n are expansion coefficients and the superscript (4) on $\mathbf{M}$ and $\mathbf{N}$ shows that these functions are constructed with spherical Bessel function of the fourth kind (or Hankel function of the second kind); i.e., $z_n(kr)$ in Eqs. (2.236) and (2.237) is replaced by $h_n^{(2)}(kr)$ when $\mathbf{M}$ and $\mathbf{N}$ are substituted in Eq. (2.238). The spherical Hankel function has been chosen to satisfy the radiation condition. In other words, the asymptotic behavior of $h_n^{(2)}(kr)$, namely,

$$h_n^{(2)}(kr) \qquad \sim \qquad j^{n+1} \frac{e^{-kr}}{kr} \, , \tag{2.239}$$

when combined with the time factor $e^{j\omega t}$, represents an outgoing spherical wave (see Eq. (2.106d)). Similarly, the transmitted field inside the sphere can be constructed as

$$\mathbf{E}^t = E_o e^{j\omega t} \sum_{n=1}^{\infty} (-j)^n \frac{2n+1}{n(n+1)} \left[c_n \mathbf{M}_n^{(1)}(k_1) + j d_n \mathbf{N}_n^{(1)}(k_1) \right] \tag{2.240a}$$

$$\mathbf{H}^t = -\frac{k E_o}{\mu\omega} e^{j\omega t} \sum_{n=1}^{\infty} (-j)^n \frac{2n+1}{n(n+1)} \left[c_n \mathbf{M}_n^{(1)}(k_1) - j d_n \mathbf{N}_n^{(1)}(k_1) \right] \tag{2.240b}$$

where c_n and d_n are expansion coefficients, k_1 is the propagation constant in the sphere. The functions $M_n^{(1)}$ and $N_n^{(1)}$ in Eq. (2.240) are obtained by replacing $z_n(kr)$ in Eq. (2.237) by $j_n(k_1 r)$; j_n is the only solution in this case since the field must be finite at the origin, the center of the sphere.

The unknown expansion coefficients a_n, b_n, c_n, and d_n are determined by letting the fields satisfy the boundary conditions, namely, the continuity of the tangential components of the total electric and magnetic fields at the surface of the sphere. Thus at $r = a$,

$$\mathbf{a}_r \times \left(\mathbf{E}^i + \mathbf{E}^s - \mathbf{E}^t \right) = 0 \tag{2.241a}$$

$$\mathbf{a}_r \times \left(\mathbf{H}^i + \mathbf{H}^s - \mathbf{H}^t \right) = 0 \tag{2.241b}$$

This is equivalent to

$$E_\theta^i + E_\theta^s = E_\theta^t, \qquad r = a \tag{2.242a}$$

$$E_\phi^i + E_\phi^s = E_\phi^t, \qquad r = a \tag{2.242b}$$

$$H_\theta^i + H_\theta^s = H_\theta^t, \qquad r = a \tag{2.242c}$$

$$H_\phi^i + H_\phi^s = H_\phi^t, \qquad r = a \tag{2.242d}$$

Substituting Eqs. (2.235), (2.238), and (2.240) into Eq. (2.242), multiplying the resulting equations by $\cos\phi$ or $\sin\phi$ and integrating over $0 \le \phi < 2\pi$, and then multiplying by $\frac{dP_m^1}{d\theta}$ or $\frac{dP_m^1}{sin\theta}$ and integrating over $0 \le \theta \le \pi$, we obtain

$$j_n(ka) + a_n h_n^{(2)}(ka) = c_n j_n\,(k_1 a) \tag{2.243a}$$

$$\mu_1\,[ka j_n(ka)]' + a_n \mu_1 \left[ka h_n^{(2)}(ka)\right]' = c_n \mu\,[k_1 a j_n\,(k_1 a)]' \tag{2.243b}$$

$$\mu_1 j_n(ka) + b_n \mu_1 h_n^{(2)}(ka) = d_n \mu j_n\,(k_1 a) \tag{2.243c}$$

$$k\,[ka j_n(ka)]' + b_n k \left[ka h_n^{(2)}(ka)\right]' = d_n k_1\,[k_1 a j_n\,(k_1 a)]' \tag{2.243d}$$

Solving Eqs. (2.243a) and (2.243b) gives a_n and c_n, while solving Eqs. (2.243c) and (2.243d) gives b_n and d_n. Thus, for $\mu = \mu_o = \mu_1$,

$$\boxed{\begin{aligned}
a_n &= \frac{j_n(m\alpha)[\alpha j_n(\alpha)]' - j_n(\alpha)[m\alpha j_n(m\alpha)]'}{j_n(m\alpha)[\alpha h_n^{(2)}(\alpha)]' - h_n^{(2)}(\alpha)[m\alpha j_n(m\alpha)]'} && (2.244a)\\
b_n &= \frac{j_n(\alpha)[m\alpha j_n(m\alpha)]' - m^2 j_n(m\alpha)[\alpha j_n(\alpha)]'}{h_n^{(2)}(\alpha)[m\alpha j_n(m\alpha)]' - m^2 j_n(m\alpha)[\alpha h_n^{(2)}(\alpha)]'} && (2.244b)\\
c_n &= \frac{j/\alpha}{h_n^{(2)}(\alpha)[m\alpha j_n(m\alpha)]' - j_n(m\alpha)[\alpha h_n^{(2)}(\alpha)]'} && (2.244c)\\
d_n &= \frac{j/\alpha}{h_n^{(2)}(\alpha)[m\alpha j_n(m\alpha)]' - m^2 j_n(m\alpha)[\alpha h_n^{(2)}(\alpha)]'} && (2.244d)
\end{aligned}}$$

where $\alpha = ka = 2\pi a/\lambda$ and $m = k_1/k$ is the refractive index of the dielectric, which may be real or complex depending on whether the dielectric is lossless or lossy. The primes at the square brackets indicate differentiation with respect to the argument of the Bessel function inside the brackets, i.e., $[xz_n(x)]' = \frac{\partial}{\partial x}[xz_n(x)]$. To obtain Eqs. (2.244c) and (2.244d), we have made use of the Wronskian relationship

$$j_n(x)\left[x h_n^{(2)}(x)\right]' - h_n^{(2)}(x)\,[x j_n(x)]' = -j/x \tag{2.245}$$

If the dielectric is lossy and its surrounding medium is free space,

$$k_1^2 = \omega\mu_o\,(\omega\epsilon_1 - j\sigma)\,, \qquad k^2 = \omega^2\mu_o\epsilon_o \tag{2.246}$$

so that the (complex) refractive index m becomes

$$m = \frac{k_1}{k} = \sqrt{\epsilon_c} = \sqrt{\epsilon_{r1} - j\frac{\sigma_1}{\omega\epsilon_o}} = m' - jm'' \tag{2.247}$$

The problem of scattering by a conducting sphere can be obtained as a special case of the problem considered above. Since the EM fields must vanish inside the conducting sphere, the right-hand sides of Eqs. (2.242a), (2.242b), (2.243a), and (2.243d) must be equal to zero so that ($c_n = 0 = d_n$)

$$a_n = -\frac{j_n(\alpha)}{h_n^{(2)}(\alpha)} \tag{2.248a}$$

$$b_n = -\frac{[\alpha j_n(\alpha)]'}{[\alpha h_n^{(2)}(\alpha)]'} \tag{2.248b}$$

Thus we have completed the Mie solution; the field at any point inside or outside the sphere can now be determined. We will now apply the solution to problems of practical interest.

2.8.2 Scattering Cross Sections

Often scattered radiation is most conveniently measured by the scattering cross section Q_{sca} (in meter^2) which may be defined as the ratio of the total energy scattered per second W_s to the energy density P of the incident wave, i.e.,

$$Q_{\text{sca}} = \frac{W_s}{P} \tag{2.249}$$

The energy density of the incident wave is given by

$$P = \frac{E_o^2}{2\eta} = \frac{1}{2}E_o^2\sqrt{\frac{\epsilon}{\mu}} \tag{2.250}$$

The scattered energy from the sphere is

$$W_s = \frac{1}{2}\,\text{Re}\int_0^{2\pi}\int_0^{\pi}\left[E_\theta H_\phi^* - E_\phi H_\theta^*\right] r^2 \sin\theta \, d\theta \, d\phi$$

where the star sign denotes complex conjugation and field components are evaluated at far field ($r >> a$). By using the asymptotic expressions for spherical Bessel functions, we can write the resulting field components as

$$E_\theta^s = \eta H_\phi^s = -\frac{j}{kr}E_o e^{j(\omega t - kr)}\cos\phi S_2(\theta) \tag{2.251a}$$

$$-E_\phi^s = \eta H_\theta^s = -\frac{j}{kr}E_o e^{j(\omega t - kr)}\sin\phi S_1(\theta) \tag{2.251b}$$

where the amplitude functions $S_1(\theta)$ and $S_2(\theta)$ are given by [11]

$$S_1(\theta) = \sum_{n=1}^{\infty} \frac{2n+1}{n(n+1)} \left(\frac{a_n}{\sin\theta} P_n^1(\cos\theta) + b_n \frac{dP_n^1(\cos\theta)}{d\theta} \right) \tag{2.252a}$$

$$S_2(\theta) = \sum_{n=1}^{\infty} \frac{2n+1}{n(n+1)} \left(\frac{b_n}{\sin\theta} P_n^1(\cos\theta) + a_n \frac{dP_n^1(\cos\theta)}{d\theta} \right) \tag{2.252b}$$

Thus,

$$W_s = \frac{\pi E_o^2}{2k^2\eta} \operatorname{Re} \int_0^{\pi} \left(|S_1(\theta)|^2 + |S_2(\theta)|^2 \right) \sin\theta \, d\theta$$

This is evaluated with the help of the identities [10]

$$\int_0^{\pi} \left(\frac{dP_n^1}{d\theta} \frac{dP_m^1}{d\theta} + \frac{1}{\sin^2\theta} P_n^1 P_m^1 \right) \sin\theta \, d\theta = \begin{cases} 0, & n \neq m \\ \dfrac{2}{2n+1} \dfrac{(n+1)!}{(n-1)} n(n+1), & n = m \end{cases}$$

and

$$\int_0^{\pi} \left(\frac{dP_m^1}{\sin\theta} \frac{dP_n^1}{d\theta} + \frac{P_n^1}{\sin\theta} \frac{P_m^1}{d\theta} \right) \sin\theta \, d\theta = 0$$

We obtain

$$W_s = \frac{\pi E_o^2}{k^2\eta} \sum_{n=1}^{\infty} (2n+1) \left(|a_n|^2 + |b_n|^2 \right) \tag{2.253}$$

Substituting Eqs. (2.250) and (2.253) into Eq. (2.249), the scattering cross section is found to be

$$Q_{\text{sca}} = \frac{2\pi}{k^2} \sum_{n=1}^{\infty} (2n+1) \left(|a_n|^2 + |b_n|^2 \right) \tag{2.254}$$

Similarly, the *cross section for extinction* Q_{ext} (in meter^2) is obtained [11] from the amplitude functions for $\theta = 0$, i.e.,

$$Q_{\text{ext}} = \frac{4\pi}{k^2} \operatorname{Re} S(0)$$

or

$$Q_{\text{ext}} = \frac{2\pi}{k^2} \operatorname{Re} \sum_{n=1}^{\infty} (2n+1) \left(a_n + b_n \right) \tag{2.255}$$

where

$$S(0) = S_1(0^o) = S_2(0^o) = \frac{1}{2}\sum_{n=1}^{\infty}(2n+1)\,(a_n + b_n) \tag{2.256}$$

In obtaining Eq. (2.256), we have made use of

$$\left.\frac{P_n^1}{\sin\theta}\right|_{\theta=0} = \left.\frac{dP_n^1}{d\theta}\right|_{\theta=0} = n(n+1)/2$$

If the sphere is absorbing, the *absorption cross section* Q_{abs} (in meter^2) is obtained from

$$Q_{\text{abs}} = Q_{\text{ext}} - Q_{\text{sca}} \tag{2.257}$$

since the energy removed is partly scattered and partly absorbed.

A useful, measurable quantity in radar communications is the *radar cross section* or *back-scattering cross section* σ_b of a scattering obstacle. It is a lump measure of the efficiency of the obstacle in scattering radiation back to the source ($\theta = 180°$). It is defined in terms of the far zone scattered field as

$$\sigma_b = 4\pi r^2 \frac{|\mathbf{E}^s|^2}{E_o^2}, \qquad \theta = \pi \tag{2.258}$$

From Eq. (2.251),

$$\sigma_b = \frac{2\pi}{k^2}\left[|S_1(\pi)|^2 + |S_2(\pi)|^2\right]$$

But

$$-S_1(\pi) = S_2(\pi) = \frac{1}{2}\sum_{n=1}^{\infty}(-1)^n(2n+1)\,(a_n - b_n)$$

where we have used

$$-\left.\frac{P_n^1}{\sin\theta}\right|_{\theta=\pi} = \left.\frac{dP_n^1}{d\theta}\right|_{\theta=\pi} = (-1)^n n(n+1)/2$$

Thus

$$\sigma_b = \frac{\pi}{k^2}\left|\sum_{n=1}^{\infty}(-1)^n(2n+1)\,(a_n - b_n)\right|^2 \tag{2.259}$$

Similarly, we may determine the *forward-scattering cross section* ($\theta = 0^o$) as

$$\sigma_f = \frac{2\pi}{k^2}\left[|S_1(0)|^2 + |S_2(0)|^2\right]$$

Substituting Eq. (2.256) into this yields

$$\sigma_f = \frac{\pi}{k^2}\left|\sum_{n=1}^{\infty}(2n+1)\,(a_n + b_n)\right|^2 \tag{2.260}$$

2.9 Attenuation Due to Raindrops

The rapid growth in demand for additional communication capacity has put pressure on engineers to develop microwave systems operating at higher frequencies. It turns out, however, that at frequencies above 10 GHz attenuation caused by atmospheric particles can reduce the reliability and performance of radar and space communication links. Such particles include oxygen, ice crystals, rain, fog, and snow. Prediction of the effect of these precipitates on the performance of a system becomes important. In this final subsection, we will examine attenuation and phase shift of an EM wave propagating through rain drops. We will assume that raindrops are spherical so that Mie rigorous solution can be applied. This assumption is valid if the rate intensity is low. For high rain intensity, an oblate spheroidal model would be more realistic [12].

The magnitude of an EM wave traveling through a homogeneous medium (with N identical spherical particles per unit volume) in a distance ℓ is given by $e^{-\gamma\ell}$, where γ is the attenuation coefficient given by [11]

$$\gamma = NQ_{\text{ext}}$$

or

$$\gamma = \frac{N\lambda^2}{\pi} \operatorname{Re} S(0) \tag{2.261}$$

Thus the wave is attenuated by

$$A = 10 \log_{10} \frac{1}{e^{-\gamma\ell}} = \gamma\ell 10 \log_{10} e$$

or

$$A = 4.343\gamma\ell \qquad \text{(in dB)}$$

The attenuation per length (in dB/m) is

$$A = 4.343\gamma$$

or

$$A = 4.343 \frac{\lambda^2 N}{\pi} \operatorname{Re} S(0) \tag{2.262}$$

Similarly, it can be shown [11] that the phase shift of the EM wave caused by the medium is

$$\Phi = -\frac{\lambda^2 N}{2\pi} \operatorname{Im} S(0) \qquad \text{(in radians/unit length)}$$

or

$$\Phi = -\frac{\lambda^2 N}{2\pi} \operatorname{Im} S(0) \frac{180}{\pi} \qquad \text{(in deg/m)} \tag{2.263}$$

Table 2.9 Laws and Parsons Drop-size Distributions for Various Rain Rates

	Rain Rate (mm/hour)								
Drop diameter (cm)	0.25	1.25	2.5	5	12.5	25	50	100	150
	Percent of total volume								
0.05	28.0	10.9	7.3	4.7	2.6	1.7	1.2	1.0	1.0
0.1	50.1	37.1	27.8	20.3	11.5	7.6	5.4	4.6	4.1
0.15	18.2	31.3	32.8	31.0	24.5	18.4	12.5	8.8	7.6
0.2	3.0	13.5	19.0	22.2	25.4	23.9	19.9	13.9	11.7
0.25	0.7	4.9	7.9	11.8	17.3	19.9	20.9	17.1	13.9
0.3		1.5	3.3	5.7	10.1	12.8	15.6	18.4	17.7
0.35		0.6	1.1	2.5	4.3	8.2	10.9	15.0	16.1
0.4		0.2	0.6	1.0	2.3	3.5	6.7	9.0	11.9
0.45			0.2	0.5	1.2	2.1	3.3	5.8	7.7
0.5				0.3	0.6	1.1	1.8	3.0	3.6
0.55					0.2	0.5	1.1	1.7	2.2
0.6						0.3	0.5	1.0	1.2
0.65							0.2	0.7	1.0
0.7									0.3

To relate attenuation and phase shift to a realistic rainfall rather than identical drops assumed so far, it is necessary to know the drop-size distribution for a given rate intensity. Representative distributions were obtained by Laws and Parsons [13] as shown in Table 2.9. To evaluate the effect of the drop-size distribution, suppose for a particular rain rate R, p is the percent of the total volume of water reaching the ground (as in Table 2.9), which consists of drops whose diameters fall in the interval centered in D cm ($D = 2a$), the number of drops in that interval is given by

$$N_c = pN(D) \tag{2.264}$$

The total attenuation and phase shift over the entire volume become

$$A = 0.4343\frac{\lambda^2}{\pi} \cdot 10^6 \sum pN(D)\, \mathrm{Re}\, S(0) \qquad \text{(dB/km)} \tag{2.265}$$

$$\Phi = -\frac{9\lambda^2}{\pi^2} \cdot 10^6 \sum pN(D)\, \mathrm{Im}\, S(0) \qquad \text{(deg/km)} \tag{2.266}$$

where λ is the wavelength in cm and $N(D)$ is the number of raindrops with equivolumic diameter D per cm^3. The summations are taken over all drop sizes. In order to relate the attenuation and phase shift to the rain intensity measured in rain rate R (in mm/hour), it is necessary to have a relationship between N and R. The relationship obtained by Best [13], shown in Table 2.10, involves the terminal velocity u (in m/s) of the rain drops, i.e.,

$$R = u \cdot N \cdot \quad \text{(volume of a drop)}$$
$$= uN\frac{4\pi a^3}{3} \quad \text{(in m/s)}$$

or

$$R = 6\pi NuD^3 \cdot 10^5 \quad \text{(mm/hr)}$$

Thus

$$N(D) = \frac{R}{6\pi uD^3}10^{-5} \tag{2.267}$$

Substituting this into Eqs. (2.265) and (2.266) leads to

$$A = 4.343\frac{\lambda^2}{\pi^2}R\sum\frac{p}{6uD^3}\,\mathrm{Re}\,S(0) \quad \text{(dB/km)} \tag{2.268}$$

$$\Phi = -90\frac{\lambda^2}{\pi^3}R\sum\frac{p}{6uD^3}\,\mathrm{Im}\,S(0) \quad \text{(deg/km)}\,, \tag{2.269}$$

where $N(D)$ is in per cm^3, D and λ are in cm, u is in m/s, p is in percent, and $S(0)$ is the complex forward-scattering amplitude defined in Eq. (2.256). The complex refractive index of raindrops [14] at 20°C required in calculating attenuation and phase shift is shown in Table 2.11.

Table 2.10 Raindrop Terminal Velocity

Radius (cm)	Velocity (m/s)
0.025	2.1
0.05	3.9
0.075	5.3
0.10	6.4
0.125	7.3
0.15	7.9
0.175	8.35
0.20	8.7
0.225	9.0
0.25	9.2
0.275	9.35
0.30	9.5
0.325	9.6

Example 2.13

For ice spheres, plot the normalized back-scattering cross section, $\sigma_b/\pi a^2$, as a function of the normalized circumference, $\alpha = 2\pi a/\lambda$. Assume that the refractive

Table 2.11 Refractive Index of Water at 20°C

Frequency (GHz)	Refractive index ($m = m' - jm''$)
0.6	$8.960 - j0.1713$
0.8	$8.956 - j0.2172$
1.0	$8.952 - j0.2648$
1.6	$8.933 - j0.4105$
2.0	$8.915 - j0.5078$
3.0	$8.858 - j0.7471$
4.0	$8.780 - j0.9771$
6.0	$8.574 - j1.399$
11	$7.884 - j2.184$
16	$7.148 - j2.614$
20	$6.614 - j2.780$
30	$5.581 - j2.848$
40	$4.886 - j2.725$
60	$4.052 - j2.393$
80	$3.581 - j2.100$
100	$3.282 - j1.864$
160	$2.820 - j1.382$
200	$2.668 - j1.174$
300	$2.481 - j0.8466$

index of ice is independent of wavelength, making the normalized cross section for ice applicable over the entire microwave region. Take $m = 1.78 - j2.4 \times 10^{-3}$ at 0°C. □

Solution

From Eq. (2.259),

$$\sigma_b = \frac{\pi}{k^2}\left|\sum_{n=1}^{\infty}(-1)^n(2n+1)\,(a_n - b_n)\right|^2$$

Since $\alpha = ka$, the normalized back-scattering cross section is

$$\frac{\sigma_b}{\pi a^2} = \frac{1}{\alpha^2}\left|\sum_{n=1}^{\infty}(-1)^n(2n+1)\,(a_n - b_n)\right|^2 \tag{2.270}$$

Using this expression in conjunction with Eq. (2.244), the subroutine SCATTERING in the FORTRAN code of Fig. 2.16 was used as the main program to determine $\sigma_b/\pi a^2$ for $0.2 < \alpha < 4$. Details on the program will be explained in the next example. It suffices to mention that the maximum number of terms of the infinite series in Eq. (2.270) was 10. It has been found that truncating the series at $n = 2\alpha$ provides sufficient accuracy. The plot of the normalized radar cross section versus α

is shown in Fig. 2.15. From the plot, we note that back-scattering oscillates between very large and small values. If α is increased further, the normalized radar cross section increases rapidly. The unexpectedly large cross sections have been attributed to a lens effect; the ice sphere acts like a lens which focuses the incoming wave on the back side from which it is reflected backwards in a concentrated beam. This is recognized as a crude description, but it at least permits visualization of a physical process which may have some reality. ∎

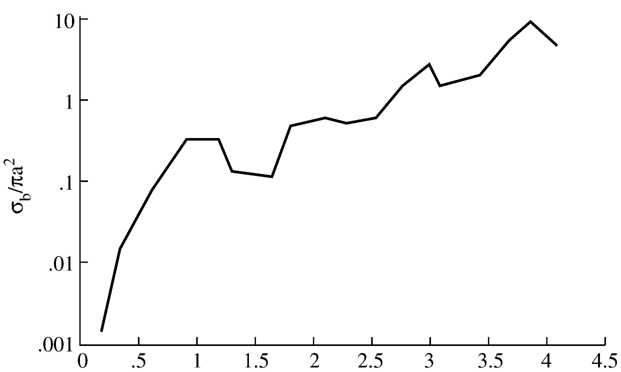


Figure 2.15
Normalized back-scattering (radar) cross sections $\alpha = 2\pi a/\lambda$ for ice at 0°C.

Example 2.14
Assuming the Laws and Parsons' rain drop-size distribution, calculate the attenuation in dB/km for rain rates of 0.25, 1.25, 2.5, 5.0, 12.5, 50.0, 100.0, and 150.0 mm/hr. Consider the incident microwave frequencies of 6, 11, and 30 GHz. □

Solution
The FORTRAN code developed for calculating attenuation and phase shift of microwaves due to rain is shown in Fig. 2.16. The main program calculates attenuation and phase shift for given values of frequency and rain rate by employing Eqs. (2.268) and (2.269). For each frequency, the corresponding value of the refractive index of water at 20°C is taken from Table 2.11. The data in Tables 2.9 and 2.10 on the drop-size distributions and terminal velocity are incorporated in the main program.

Seven subroutines are involved. The subroutine SCATTERING calculates the expansion coefficients a_n, b_n, c_n and d_n using Eq. (2.244) and also the forward-scattering amplitude $S(0)$ using Eq. (2.256). The same subroutine was used as the main program in the previous example to calculate the radar cross section of ice spheres. Enough comments are inserted to make the program self-explanatory. Subroutine BESSEL and BESSELCMPLX are exactly the same except that the former is for real argument, while the latter is for complex argument. They both employ Eq. (2.160) to

```
C ******************************************************
C  MAIN PROGRAM
C
C  FOR SPHERICAL RAIN DROPS,
C  THIS PROGRAM CALCULATES ATTENUATION IN dB/KM AND
C  PHASE SHIFT IN DEG/KM FOR A GIVEN RAIN RATE
C
C  R = RAIN RATE IN MM/HR
C  D = DROP DIAMETER IN CM
C  F = FREQUENCY IN GHZ
C  AT = ATTENUATION IN dB/KM
C  PH = PHASE SHIFT IN DEG/KM
C  V = TERMINAL VELOCITY OF RAIN DROPS
C  P = PERCENT OF TOTAL VOLUME AS MEASURED
C  BY LAWS AND PARSONS
C  M = COMPLEX REFRACTIVE INDEX OF WATER AT T = 20 C
C  X = ALPHA = K*A

       DIMENSION R(9),P(14,9),V(14)
       REAL*8 LAM,MR,MI
       COMPLEX*8 M,S0,S00(15)
       DATA V/2.1,3.9,5.3,6.4,7.3,7.9,8.35,8.7,9.0,9.2,
     1        9.35,9.5,9.6,9.6/
       DATA R/0.25,1.25,2.5,5.0,12.5,25.0,50.0,100.0,150.0/
       DATA (( P(I,J),I=1,14),J=1,9) /28.0,50.1,18.2,3.0,0.7,
     1        9*0.0,  10.9,37.1,31.3,13.5,4.9,1.5,0.6,0.2,6*0.0,
     1        7.3,27.8,32.8,19.0,7.9,3.3,1.1,0.6,0.2,5*0.0,
     2        4.7,20.3,31.0,22.2,11.8,5.7,2.5,1.0,0.5,0.3,4*0.0,
     3        2.6,11.5,24.5,25.4,17.3,10.1,4.3,2.3,1.2,0.6,
     4        0.2,3*0.0,  1.7,7.6,18.4,23.9,19.9,12.8,8.2,3.5,2.1,
     5        1.1,0.5,0.3,2*0.0,  1.2,5.4,12.5,19.9,20.9,15.6,
     6        10.9,6.7,3.3,1.8,1.1,0.5,0.2,0.0,  1.0,4.6,8.8,
     7        13.9,17.1,18.4,15.0,9.0,5.8,3.0,1.7,1.0,0.7,0.0,
     8        1.0,4.1,7.6,11.7,13.9,17.7,16.1,11.9,7.7,3.6,2.2,
     9        1.2,1.0,0.3 /

        F = 30.0
        LAM = 30.0/F        ! WAVELENGTH IN CM
        MR = 5.581
        MI = 2.848
        M = CMPLX(MR,-MI)
        PIE = 3.141592654
        DO  10  I = 1,14
        D = 0.05*DFLOAT(I)
        X = PIE*D/LAM
        CALL SCATTERING(X,M,S0,SB)
        S00(I) = S0
10      CONTINUE
        DO 50 N = 1,9
        SUMAT = 0.0
        SUMPH = 0.0
        WRITE(6,20) R(N),F
20       FORMAT(5X,'RAIN RATE =',F8.4,1X,'MM/HR',4X
     1             'FREQUENCY =',F10.5,1X,'GHZ',/)
      DO 30 K=1,14
      D = 0.05*DFLOAT(K)
      X = PIE*D/LAM
      S0 = S00(K)
C
```

Figure 2.16

FORTRAN program for Examples 2.13 and 2.14 ***(Continued).***

```
C       CALCULATE ATTENUATION AND PHASE SHIFT
C
        ACO = REAL(SO)/(6.0*X*X)
        PCO =  AIMAG(SO)/(12.0*X*X)
        ATO = ACO*( P(K,N)/100.0 )/ (V(K)*D )
        SUMAT = SUMAT + ATO
        PHO = PCO*( P(K,N)/100.0 )/ (V(K)*D )
        SUMPH = SUMPH + PHO
 30     CONTINUE
        AT = 4.343*R(N)*SUMAT
        PH = 180.0*R(N)*SUMPH/PIE
        PRINT *,R(N),AT,PH
        WRITE(6,40) AT,PH
 40     FORMAT(3X,'ATTENUATION =',F12.6,1X,'dB/KM',3X,
     1             'PHASE SHIFT =',F12.6,1X,'DEG/KM',/)
 50    CONTINUE
 60      FORMAT(I5,F15.8,F15.8)
         STOP
         END

C ********************************************************
C        USING MIE'S SOLUTION,
C        THIS SUBROUTINE CALCULATES THE SCATTERING COEFFIENTS
C        AND THE FORWARD SCATTERING FUNCTION
C
         SUBROUTINE SCATTERING(X,M,SO,SB)

         IMPLICIT COMPLEX*8 (A-D)
         COMPLEX*8 M,SO,Y,JC,SUM,SUMB
         COMPLEX*8 JM(0:20),JMD(0:20),H(0:20),HD(0:20)
         REAL*8  J(0:20),JD(0:20)
         DIMENSION A(20),B(20),C(20),D(20)

         Y = M*X
         JC=(0.0,1.0)
         SUM = (0.0,0.0)
         SUMB = (0.0,0.0)
         NMAX = 10
C
C        FIRST, CALCULATE THE SCATTERING COEFFICIENTS an and bn
C
         DO 10 N=1,NMAX
         CALL BESSEL(X,N,J,JD)
         CALL BESSELCMPLX(Y,N,JM,JMD)
         CALL HANKEL(X,N,H,HD)
         A1 = JM(N)*JD(N) - J(N)*JMD(N)
         A2 = JM(N)*HD(N) - H(N)*JMD(N)
         A(N) =  A1/A2
         B1 = J(N)*JMD(N) - (M**2)*JM(N)*JD(N)
         B2 = H(N)*JMD(N) - (M**2)*JM(N)*HD(N)
         B(N) =  B1/B2
         C(N) = JC/(X*A2)
         D(N) = - JC*M/(X*B2)
C
C       CALCULATE THE FORWARD SCATTERING AMPLITUDE FUNCTION S(0)
C
        F = 2.0*FLOAT(N) + 1.0
        SUM = SUM + F*( A(N) + B(N) )
        SUMB = SUMB + F*((-1.)**N)*( A(N) - B(N) )
 10     CONTINUE
```

Figure 2.16

(Cont.) **FORTRAN program for Examples 2.13 and 2.14** ***(Continued).***

```
      SO = SUM/2.0        ! FORWARD SCATTERING AMPLITUDE FUNCTION
      SB=(CABS( SUMB )/X)**2  ! NORMALIZED RADAR CROSS-SECTION
      RETURN
      END
```

```
C ************************************************************
C     SUBROUTINE FOR SPHERICAL HANKEL FUNCTIONS H AND HD
C     OF REAL ARGUMENT (SERIES EXPANSION METHOD)

      SUBROUTINE HANKEL(X,N,H,HD)
      IMPLICIT REAL*8(A-G,J,O-Z),COMPLEX*8(H)
      DIMENSION J(0:20),JD(0:20),Y(0:20),YD(0:20)
      DIMENSION JN(0:20),JND(0:20),H(0:20),HD(0:20)

      PIE = 3.141592654
      CALL BESSEL(X,N,J,JD)
      CALL BESSELN(X,N,JN,JND)
      V = DFLOAT(N) + 0.5
      Y(N) = ( DCOS(V*PIE)*J(N) - JN(N) )/DSIN(V*PIE)
      H(N) = CMPLX( J(N), - Y(N) )      ! HANKEL OF 2ND KIND
      YD(N) = ( DCOS(V*PIE)*JD(N) - JND(N) )/DSIN(V*PIE)
      HD(N) = CMPLX( JD(N), - YD(N) )
      RETURN
      END
```

```
C ************************************************************
C     SUBROUTINE FOR SPHERICAL BESSEL FUNCTION J AND JD
C     OF REAL ARGUMENT (SERIES EXPANSION METHOD)
C
      SUBROUTINE BESSEL(X,N,J,JD)
      IMPLICIT REAL*8(A-H,J,O-Z)
      DIMENSION J(0:20),JD(0:20)

      TOL = 0.00000001          ! TOLERANCE
      PIE = 3.141592654
      V = DFLOAT(N) + 0.5
      K = 0
      SUM1 = 0.0
      SUM2 = 0.0
10    PN = V + DFLOAT(K) + 1.0
      CALL GAMMA(PN,GN)
      CALL FACTORIAL(K,FK)
      A = ((-1.)**K)*((.5)**(V+2*K))*(X**(V+2*K-.5))/(GN*FK)
      SUM1 = SUM1 + A
      B = A*(V + FLOAT(2*K) + 0.5)
      SUM2 = SUM2 + B
      K = K + 1
      IF(ABS(A).GE.TOL) GO TO 10
      CONTINUE
      Q = DSQRT(PIE/2.0)
      J(N) = Q*SUM1
      JD(N) = Q*SUM2
      RETURN
      END
```

```
C ************************************************************
C     SUBROUTINE FOR SPHERICAL BESSEL FUNCTIONS JN AND JND
C     OF REAL ARGUMENT BUT NEGATIVE ORDER (SERIES EXPANSION)
C
```

Figure 2.16

(Cont.) **FORTRAN program for Examples 2.13 and 2.14** ***(Continued)*****.**

```
      SUBROUTINE BESSELN(X,N,JN,JND)
      IMPLICIT REAL*8(A-H,J,O-Z)
      DIMENSION JN(0:20),JND(0:20)

      TOL = 0.00000001          ! TOLERANCE
      PIE = 3.141592654
      V = DFLOAT(N) + 0.5
      K = 0
      SUM1 = 0.0
      SUM2 = 0.0
10    PN = -V + DFLOAT(K) + 1.0
      CALL GAMMA(PN,GN)
      CALL FACTORIAL(K,FK)
      A = ((-1.)**K)*((.5)**(-V+2*K))*(X**(-V+2*K-.5))/(GN*FK)
      SUM1 = SUM1 + A
      B = A*(-V + FLOAT(2*K) + 0.5)
      SUM2 = SUM2 + B
      K = K + 1
      IF(ABS(A).GE.TOL) GO TO 10
      CONTINUE
      Q = DSQRT(PIE/2.0)
      JN(N) = Q*SUM1
      JND(N) = Q*SUM2
      RETURN
      END

C ********************************************************
C     SUBROUTINE FOR SPHERICAL BESSEL FUNCTIONS JM AND JMD
C     OF COMPLEX ARGUMENT  (SERIES EXPANSION)
C
      SUBROUTINE BESSELCMPLX(Z,N,JM,JMD)
      IMPLICIT COMPLEX*8(A-D,J,S,Z),REAL*8(G,P-R,V)
      DIMENSION JM(0:20),JMD(0:20)

      TOL = 0.001          ! TOLERANCE
      PIE = 3.141592654
      V = DFLOAT(N) + 0.5
      K = 0
      SUM1 = (0.0,0.0)
      SUM2 = (0.0,0.0)
10    PN = V + DFLOAT(K) + 1.0
      CALL GAMMA(PN,GN)
      CALL FACTORIAL(K,FK)
      A = ((-1.)**K)*((.5)**(V+2*K))*(Z**(V+2*K-.5))/(GN*FK)
      SUM1 = SUM1 + A
      B = A*(V + FLOAT(2*K) + 0.5)
      SUM2 = SUM2 + B
      K = K + 1
      IF( CABS(A).GE.TOL) GO TO 10
      CONTINUE
      Q = DSQRT(PIE/2.0)
      JM(N) = Q*SUM1
      JMD(N) = Q*SUM2
      RETURN
      END

C ********************************************************
C    SUBROUTINE FOR GAMMA FUNCTION
C
```

Figure 2.16

(Cont.) **FORTRAN program for Examples 2.13 and 2.14 *(Continued)*.**

```
      SUBROUTINE GAMMA(V,G)
      IMPLICIT REAL*8(A-H,O-Z)

      PIE = 3.1415927
      IF(V-0.5) 10,20,20
10    N = -V + 0.5
      N2 = 2*N
      CALL FACTORIAL(N,FN)
      CALL FACTORIAL(N2,FN2)
      G = ((-4.)**N)*DSQRT(PIE)*FN/FN2
      RETURN
20    N = V - 0.5
      N2 = 2*N
      CALL FACTORIAL(N,FN)
      CALL FACTORIAL(N2,FN2)
      G = FN2*DSQRT(PIE)/(FN*(2.**N2))
      RETURN
      END
```

```
C **********************************************************
C     SUBROUTINE FOR FACTORIAL OF N, i.e. N!
C
      SUBROUTINE FACTORIAL(N,F)
      REAL*8 F

      F = 1.0
      IF(N.EQ.0) GO TO 20
      DO 10 I=1,N
10    F = F*DFLOAT(I)
20    RETURN
      END
```

Figure 2.16

***(Cont.)* FORTRAN program for Examples 2.13 and 2.14.**

find $j_n(x)$. Subroutine HANKEL employs Eq. (2.162) to find $y_n(x)$, which involves calling subroutine BESSELN to calculate $j_{-n}(x)$. The derivative of Bessel-Riccati function $[xz_n(x)]$ is obtained from (see Prob. 2.14)

$$[xz_n(x)]' = -nz_n(x) + xz_{n-1}(x)$$

where z_n is j_n, j_{-n}, y_n or $h_n(x)$. Subroutine GAMMA calculates $\Gamma(n + 1/2)$ using Eq. (2.165), while subroutine FACTORIAL determines $n!$. All computations were done in double precision arithmetic, although it was observed that using single precision would only alter results slightly.

Typical results for 11 GHz are tabulated in Table 2.12. A graph of attenuation vs. rain rate is portrayed in Fig. 2.17. The plot shows that attenuation increases with rain rate and conforms with the common rule of thumb. We must note that the underlying assumption of spherical raindrops renders the result as only a first order approximation of the practical rainfall situation. ∎

Table 2.12 Attenuation and Phase Shift at 11 GHz

Rain rate (mm/hr)	Attenuation (dB/km)	Phase shift (deg/km)
0.25	2.56×10^{-3}	0.4119
1.25	1.702×10^{-3}	1.655
2.5	4.072×10^{-3}	3.040
5.0	9.878×10^{-3}	5.601
12.5	0.3155	12.58
25	0.7513	23.19
50	1.740	42.74
100	3.947	78.59
150	6.189	112.16

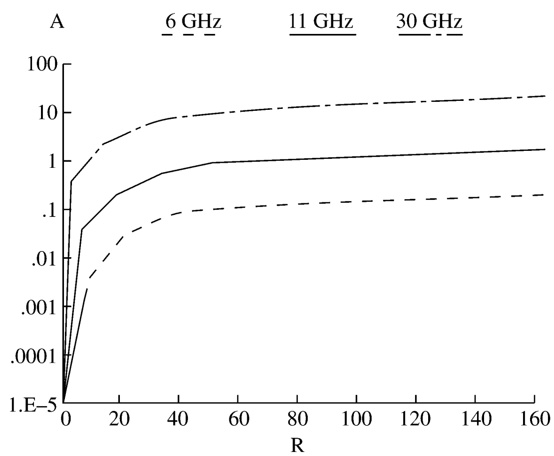


Figure 2.17
Attenuation vs. rain rate.

2.10 Concluding Remarks

We have reviewed analytic methods for solving partial differential equations. Analytic solutions are of major interest as test models for comparison with numerical techniques. The emphasis has been on the method of separation of variables, the most powerful analytic method. For an excellent, more in-depth exposition of this method, consult Myint-U [5]. In the course of applying the method of separation of variables, we have encountered some mathematical functions such as Bessel functions and Legendre polynomials. For a thorough treatment of these functions and their properties, Bell [7] and Johnson and Johnson [8] are recommended. The mathematical handbook by Abramowitz and Stegun [15] provides tabulated values of these functions for specific orders and arguments. A few useful texts on the topics covered in this chapter are also listed in the references.

As an example of real life problems, we have applied the analytical techniques developed in this chapter to the problem of attenuation of microwaves due to spherical raindrops. Spherical models have also been used to assess the absorption characteristics of the human skull exposed to EM plane waves [16]–[20] (see Probs. 2.46 to 2.49).

We conclude this chapter by remarking that the most satisfactory solution of a field problem is an exact analytical one. In many practical situations, no solution can be obtained by the analytical methods now available, and one must therefore resort to numerical approximation, graphical or experimental solutions. (Experimental solutions are usually very expensive, while graphical solutions are not so accurate). The remainder of this book will be devoted to a study of the numerical methods commonly used in EM.

References

[1] W.J. Gibbs, *Conformal Transformation in Electrical Engineering*. London: Chapman & Hall, 1958.

[2] N. Morita et al., *Integral Equation Methods for Electromagnetics*. Boston, MA: Artech House, 1990.

[3] H.F. Weinberger, *A First Course in Partial Differential Equations*. New York: John Wiley, 1965, Chap. IV, pp. 63–116.

[4] R.D. Kersten, *Engineering Differential Systems*. New York: McGraw-Hill, 1969, Chap. 5, pp. 66–106.

[5] T. Myint-U, *Partial Differential Equations of Mathematical Physics*. New York: North-Holland, 1980.

[6] G.N. Watson, *Theory of Bessel Functions*. London: Cambridge University Press, 1966.

[7] W.W. Bell, *Special Functions for Scientists and Engineers*. London: D. Van Nostrand, 1968.

[8] D.E. Johnson and J.R. Johnson, *Mathematical Methods in Engineering and Physics*. Englewood Cliffs, NJ: Prentice-Hall, 1982.

[9] K. Asano, "Electrostatic potential and field in a cylindrical tank containing charged liquid," *Proc. IEEE,* vol. 124, no. 12, Dec. 1977, pp. 1277–1281.

[10] J.A. Stratton, *Electromagnetic Theory.* New York: McGraw-Hill, 1941, pp. 394–421, 563–573.

[11] H.C. Van de Hulst, *Light Scattering of Small Particles*. New York: John Wiley, 1957, pp. 28–37, 114–136, 284.

[12] J. Morrison and M.J. Cross, "Scattering of a plane electromagnetic wave by axisymmetric raindrops," *Bell Syst. Tech. J.,* vol. 53, no. 6, July-Aug. 1974, pp. 955–1019.

[13] D.E. Setzer, "Computed transmission through rain at microwave and visible frequencies," *Bell Syst. Tech. J.,* vol. 49, no. 8, Oct. 1970, pp. 1873–1892.

[14] M.N.O. Sadiku, "Refractive index of snow at microwave frequencies," *Appl. Optics,* vol. 24, no. 4, Feb. 1985, pp. 572–575.

[15] M. Abramowitz and I.A. Stegun, *Handbook of Mathematical Functions*. New York: Dover, 1965.

[16] C.H. Durney, "Electromagnetic dosimetry for models of humans and animals: a review of theoretical and numerical techniques," *Proc. IEEE.,* vol. 68, no. 1, Jan. 1980, pp. 33–40.

[17] M.A. Morgan, "Finite element calculation of microwave absorption by the cranial structure," *IEEE Trans. Bio. Engr.,* vol. BME-28, no. 10, Oct. 1981, pp. 687–695.

[18] J.W. Hand, "Microwave heating patterns in simple tissue models," *Phys. Med. Biol.,* vol. 22, no. 5, 1977, pp. 981–987.

[19] W.T. Joines and R.J. Spiegel, "Resonance absorption of microwaves by the human skull," *IEEE Trans. Bio. Engr.,* vol. BME-21, Jan. 1974, pp. 46–48.

[20] C.M. Weil, "Absorption characteristics of multilayered sphere models exposed to UHF/microwave radiation," *IEEE Trans. Bio. Engr.,* vol. BME-22, no. 6, Nov. 1975, pp. 468–476.

Problems

2.1 Consider the PDE

$$a\Phi_{xx} + b\Phi_{xy} + c\Phi_{yy} + d\Phi_x + e\Phi_y + f\Phi = 0$$

where the coefficients a, b, c, d, e, and f are in general functions of x and y. Under what conditions is the PDE separable?

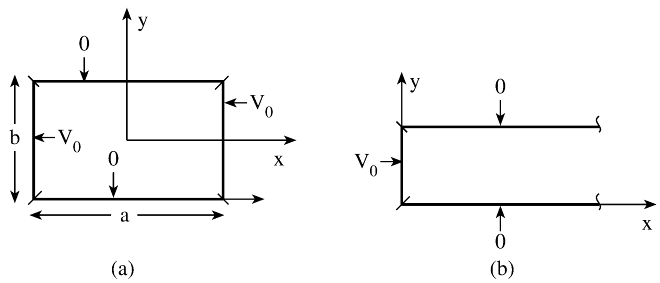


Figure 2.18
For problem 2.2.

2.2 Determine the distribution of electrostatic potential inside the conducting rectangular boxes with cross sections shown in Fig. 2.18.

2.3 The cross-sections of the cylindrical systems that extend to infinity in the z-direction are shown in Fig. 2.19. The potentials on the boundaries are as shown. For each system, find the potential distribution.

2.4 Find the solution U of:

(a) Laplace equation

$$\begin{aligned}\nabla^2 U = 0, \qquad 0 < x, y < \pi\\ U_x(0, y) = 0 = U_x(x, y), \qquad U(x, 0) = 0\,,\\ U(x, \pi) = x, \qquad 0 < x < \pi\end{aligned}$$

(b) Heat equation

$$\begin{aligned}kU_{xx} = U_t, \qquad 0 \le x \le 1,\ t > 0\\ U(0, t) = 0,\ t > 0, \qquad U(1, t) = 1,\ t > 0\\ U(x, 0) = 0, \qquad 0 \le x \le 1\end{aligned}$$

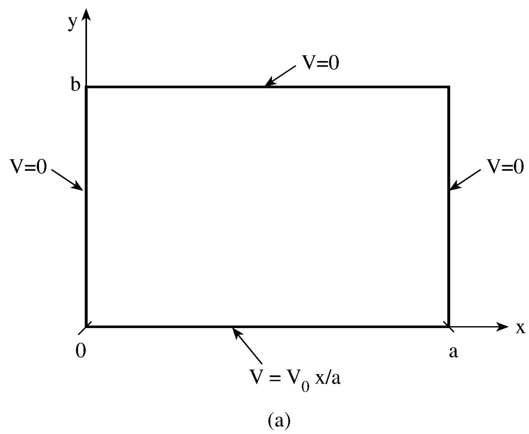
(c) Wave equation

$$\begin{aligned}a^2 U_{xx} = U_{tt}, \qquad 0 \le x \le 1,\ t > 0\\ U(0, t) = 0 = U(1, t),\ t > 0\\ U(x, 0) = 0, \qquad U_t(x, 0) = x\end{aligned}$$

2.5 Find the solution Φ of:

(a) Laplace equation

$$\begin{aligned}\nabla^2 \Phi = 0, \qquad \rho \ge 1,\ 0 < \phi < \pi\\ \Phi(1, \phi) = \sin\phi, \qquad \Phi(\rho, 0) = \Phi(\rho, \pi) = 0\end{aligned}$$



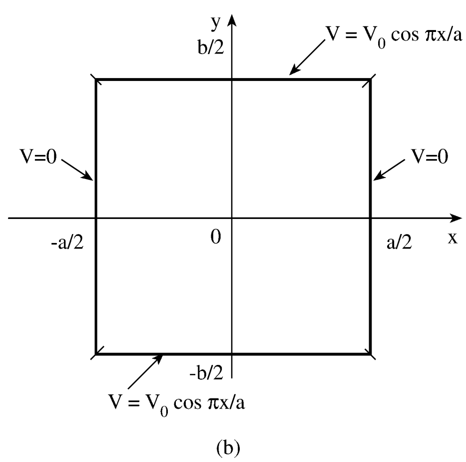


Figure 2.19
For problem 2.3.

(b) Laplace equation

$$\nabla^2\Phi = 0, \qquad 0 < \rho < 1,\ 0 < z < L$$
$$\Phi(\rho, \phi, 0) = 0 = \Phi(\rho, \phi, L), \Phi(a, \phi, z) = 1$$

(c) Heat equation

$$\Phi_t = k\nabla^2\Phi, \qquad 0 \le \rho \le 1, -\infty < z < \infty,\ t > 0$$
$$\Phi(a, \phi, t) = 0,\ t > 0, \qquad \Phi(\rho, \phi, 0) = \rho^2 \cos 2\phi, 0 \le \phi < 2\pi$$

2.6 Solve the PDE

$$4\frac{\partial^4\Phi}{\partial x^4}+\frac{\partial^2\Phi}{\partial t^2}=0, \qquad 0<x<1,\ t>0$$

subject to the boundary conditions

$$\Phi(0,t)=0=\Phi(1,t)=\Phi_{xx}(0,t)=\Phi_{xx}(1,t)$$

and initial conditions

$$\Phi_t(x,0)=0, \qquad \Phi(x,0)=x$$

2.7 A cylinder similar to the one in Fig. 2.20 has its ends $z=0$ and $z=L$ held at zero potential. If

$$V(a,z)=\begin{cases} V_o z/L, & 0<z<L/2 \\ V_o(1-z/L), & L/2<z<L \end{cases}$$

find $V(\rho,z)$. Calculate the potential at $(\rho,z)=(0.8a,0.3L)$.

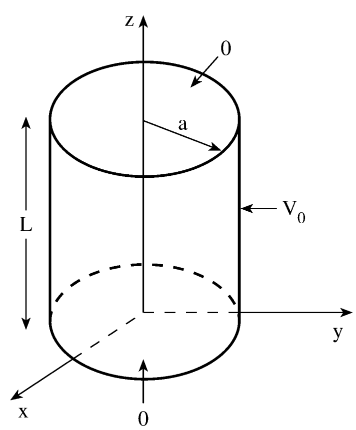


Figure 2.20
For problem 2.7.

2.8 Determine the potential distribution in a hollow cylinder of radius a and length L with ends held at zero potential while the lateral surface is held at potential V_o as in Fig. 2.20. Calculate the potential along the axis of the cylinder when $L=2a$.

2.9 The conductor whose cross section is shown in Fig. 2.21 is maintained at $V=0$ everywhere except on the curved electrode where it is held at $V=V_o$. Find the potential distribution $V(\rho,\phi)$.

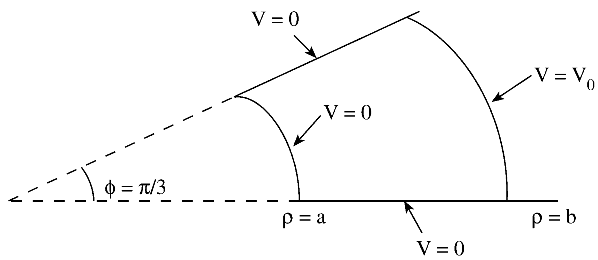


Figure 2.21
For problem 2.9.

2.10 Solve the PDE

$$\frac{\partial^2 \Phi}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial \Phi}{\partial \rho} = \frac{\partial^2 \Phi}{\partial t^2}, \qquad 0 \le \rho \le a,\ t \ge 0$$

under the conditions

$$\Phi(0,t) \quad \text{is bounded}, \qquad \Phi(a,t) = 0,\ t \ge 0\,,$$
$$\Phi(\rho,0) = \left(1 - \rho^2/a^2\right), \qquad \left.\frac{\partial \Phi}{\partial t}\right|_{t=0} = 0,\ \ 0 \le \rho \le a$$

2.11 The generating function for Bessel function is given by

$$G(x,t) = \exp\left[\frac{x}{2}\left(t - \frac{1}{t}\right)\right] = \sum_{n=-\infty}^{\infty} t^n J_n(x)$$

(a) By taking the derivatives of both sides with respect to x, show that

$$\frac{d}{dx} J_n(x) = \frac{1}{2}\left[J_{n-1}(x) - J_{n+1}(x)\right]$$

(b) By taking the derivatives of both sides with respect to t, show that

$$J_{n+1}(x) = \frac{x}{2(n+1)}\left[J_n(x) + J_{n+2}(x)\right]$$

2.12 (a) Prove that

$$e^{\pm j\rho\sin\phi} = \sum_{n=-\infty}^{\infty} (\pm 1)^n J_n(\rho) e^{jn\phi}$$

(b) Derive the *Bessel's integral formula*

$$J_n(\rho) = \frac{1}{\pi}\int_0^{\pi} \cos(n\theta - \rho\sin\theta)\, d\theta$$

2.13 Show that

$$\cos x = J_o(x) + 2\sum_{n=1}^{\infty}(-1)^n J_{2n}(x)$$

and

$$\sin x = 2\sum_{n=1}^{\infty}(-1)^{n+1} J_{2n+1}(x)$$

which demonstrate the close tie between Bessel function and trigonometric functions.

2.14 Prove that:

(a) $J_{1/2}(x) = \sqrt{\frac{2}{\pi x}}\sin x,$

(b) $J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}}\cos x,$

(c) $\frac{d}{dx}[x^{-n}J_n(x)] = -x^n J_{n+1}(x).$

(d) $\frac{d^n}{dx^n}J_n(x)\Big|_{x=0} = \frac{1}{2^n},$

(e) $\frac{d}{dx}[xz_n(x)] = -nz_n(x) + xz_{n-1}(x) = (n+1)z_n(x) + xz_{n+1}(x)$

2.15 Given that

$$I_o = \int_0^\infty e^{-\lambda a} J_o(\lambda\rho)\, d\lambda = \frac{1}{(\rho^2 + a^2)^{1/2}}$$

find

$$I_1 = \int_0^\infty e^{-\lambda a}\lambda J_o(\lambda\rho)\, d\lambda$$

and

$$I_2 = \int_0^\infty e^{-\lambda a}\lambda^2 J_o(\lambda\rho)\, d\lambda$$

2.16 Write a computer program that will evaluate the first five roots λ_{nm} of Bessel function $J_n(x)$ for $n = 1, 2, \ldots, 5$, i.e., $J_n(\lambda_{nm}) = 0$.

2.17 Evaluate:

(a)

$$\int_{-1}^{1} P_1(x)P_2(x)\, dx\ ,$$

(b)

$$\int_{-1}^{1} [P_4(x)]^2\ dx\ ,$$

(c)

$$\int_0^1 x^2 P_3(x)\, dx$$

2.18 In Legendre series of the form $\sum_{n=0}^{\infty} A_n P_n(x)$, expand:

(a) $f(x) = \begin{cases} 0, & -1 < x < 0, \\ 1, & 0 < x < 1 \end{cases}$

(b) $f(x) = x^3, \qquad -1 < x < 1,$

(c) $f(x) = \begin{cases} 0, & -1 < x < 0, \\ x, & 0 < x < 1 \end{cases}$

(d) $f(x) = \begin{cases} 1 + x, & -1 < x < 0, \\ 1 - x, & 0 < x < 1 \end{cases}$

2.19 Solve Laplace's equation:

(a) $\nabla^2 U = 0, \qquad 0 \le r \le a, \qquad U(a, \theta) = \begin{cases} 1, & 0 < \theta < \pi/2, \\ 0, & \text{otherwise} \end{cases}$

(b) $\nabla^2 U = 0, \qquad r > a, \qquad \left.\frac{\partial U}{\partial r}\right|_{r=a} = \cos\theta + 3\cos^3\theta, \quad 0 < \theta < \pi,$

(c) $\nabla^2 U = 0, \qquad r < a, \quad 0 < \theta < \pi, \quad 0 < \phi < 2\pi,$
$U(a, \theta, \phi) = \sin^2\theta$

2.20 A hollow conducting sphere of radius a has its upper half charged to potential V_o while its lower half is grounded. Find the potential distribution inside and outside the sphere.

2.21 A circular disk of radius a carries charge of surface charge density ρ_o. Show that the potential at point $(0, 0, z)$ on its axis $\theta = 0$ is

$$V = \frac{\rho_o}{2\epsilon}\left[\left(z^2 + a^2\right)^{1/2} - z\right]$$

From this deduce the potential at any point (r, θ, ϕ).

2.22 (a) Verify the three-term recurrence relation

$$(2n + 1)x P_n(x) = (n + 1)P_{n+1}(x) + n P_{n-1}(x)$$

(b) Use the recurrence relation to find $P_6(x)$ and $P_1(x)$.

2.23 Verify the following identities:

(a) $\int_{-1}^{1} P_n(x) P_m(x)\, dx = \frac{2}{2n + 1}\delta_{nm},$

(b) $\int_{-1}^{1} P_n^m(x) P_k^m(x)\, dx = \frac{2}{2n + 1}\frac{(n + m)!}{(n - m)!}\delta_{nk}$

2.24 Rework the problem in Fig. 2.8 if the boundary conditions are now

$$V(r=a)=V_o, \qquad V(r\to\infty)=E_o r\cos\theta+V_o$$

Find V and $\mathbf{E}$ everywhere. Determine the maximum value of the field strength.

2.25 In a sphere of radius a, obtain the solution $V(r,\theta)$ of Laplace's equation

$$\nabla^2 V(r,\theta)=0, \qquad r\le a$$

subject to

$$V(a,\theta)=3\cos^2\theta+3\cos\theta+1$$

2.26 Determine the solution to Laplace's equation

$$\nabla^2 V=0$$

outside a sphere $r>a$ subject to the boundary condition

$$\frac{\partial}{\partial r}V(a,\theta)=\cos\theta+3\cos^3\theta$$

2.27 Find the potential distribution inside and outside a dielectric sphere of radius a placed in a uniform electric field E_o.

Hint: The problem to be solved is $\nabla^2 V=0$ subject to

$$\begin{aligned}\epsilon_r\frac{\partial V_1}{\partial r}&=\frac{\partial V_2}{\partial r} \qquad \text{on } r=a,\ V_1=V_2 \text{ on } r=a\,,\\ V_2&=-E_o r\cos\theta \qquad \text{as } r\to\infty\end{aligned}$$

2.28 (a) Derive the recurrence relation of the associated Legendre polynomials

$$P_n^{m+1}(x)=\frac{2mx}{(1-x^2)^{1/2}}P_n^m(x)-[n(n+1)-m(m-1)]P_n^{m-1}(x)$$

(b) Using the recurrence relation on the formula for P_n^m, find P_3^2, P_3^3, P_4^1, and P_4^2.

2.29 Expand $V=\cos 2\phi\sin^2\phi$ in terms of the spherical harmonics $P_n^m(\cos\theta)\sin m\phi$ and $P_n^m(\cos\theta)\cos m\phi$.

2.30 In the prolate spheroidal coordinates (ξ,η,ϕ), the equation

$$\nabla^2\Phi+k^2\Phi=0$$

assumes the form

$$\begin{aligned}\frac{\partial}{\partial\xi}\left[\left(\xi^2-1\right)\frac{\partial\Phi}{\partial\xi}\right]+\frac{\partial}{\partial\eta}\left[\left(1-\eta^2\right)\frac{\partial\Phi}{\partial\eta}\right]+\left[\frac{1}{\xi^2-1}\right.\\ \left.+\frac{1}{1-\eta^2}\right]\frac{\partial^2\Phi}{\partial\phi^2}+k^2d^2\left(\xi^2-\eta^2\right)\Phi=0\end{aligned}$$

Show that the separated equations are

$$\frac{d}{d\xi}\left[\left(\xi^2+1\right)\frac{d\Psi_1}{d\xi}\right]+\left[k^2d^2\xi^2-\frac{m^2}{\xi^2-1}-c\right]\Psi_1=0$$

$$\frac{d}{d\eta}\left[\left(1-\eta^2\right)\frac{d\Psi_2}{d\eta}\right]-\left[k^2d^2\eta^2+\frac{m^2}{1-\eta^2}-c\right]\Psi_2=0$$

$$\frac{d^2\Psi_3}{d\phi^2}+m^2\Psi_3=0$$

where m and c are separation constants.

2.31 Solve Eq. (2.203) if $a=b=c=\pi$ and:
(a) $f(x,y,z)=e^{-x}$, (b) $f(x,y,z)=\sin^2 x$.

2.32 Solve the inhomogeneous PDE

$$\frac{\partial^2\Phi}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial\Phi}{\partial\rho}-\frac{\partial^2\Phi}{\partial t^2}=-\Phi_o\sin\omega t,\qquad 0\le\rho\le a,\ t\ge 0$$

subject to the conditions $\Phi(a,t)=0$, $\Phi(\rho,0)=0$, $\Phi_t(\rho,0)=0$, Φ is finite for all $0\le\rho\le a$. Take Φ_o as a constant and $a\omega$ not being a zero of $J_0(x)$.

2.33 Infinitely long metal box has a rectangular cross section shown in Fig. 2.22. If the box is filled with charge $\rho_v=\rho_o x/a$, find V inside the box.

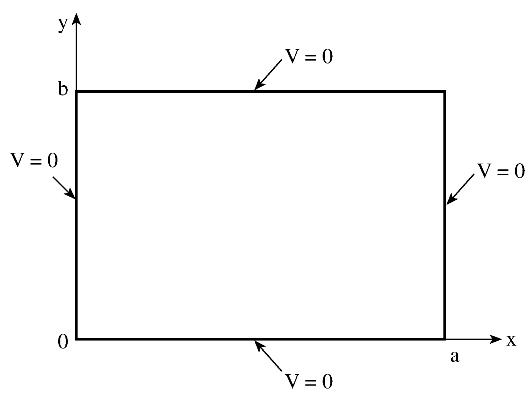


Figure 2.22
For problem 2.33.

2.34 In Section 2.7.2, find $\mathbf{E}_g$ and $\mathbf{E}_\ell$, the electric field intensities in gas and liquid, respectively.

2.35 Consider the potential problem shown in Fig. 2.23. The potentials at $x=0$, $x=a$, and $y=0$ sides are zero while the potential at $y=b$ side is V_o. Using the series expansion technique similar to that used in Section 2.7.2, find the potential distribution in the solution region.

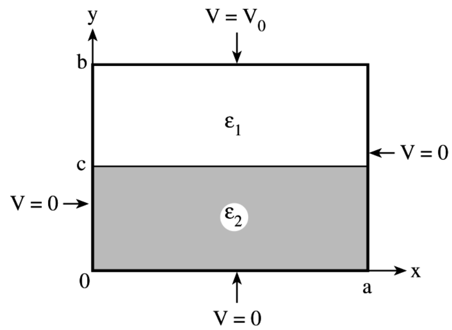


Figure 2.23
Potential system for problem 2.35.

2.36 Consider a grounded rectangular pipe with the cross section shown in Fig. 2.24. Assuming that the pipe is partially filled with hydrocarbons with charge density ρ_o, apply the same series expansion technique used in Section 2.7.2 to find the potential distribution in the pipe.

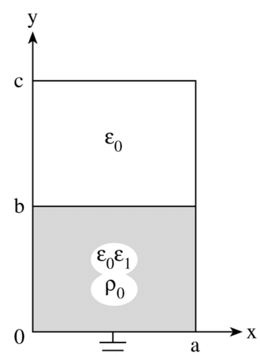


Figure 2.24
Earthed rectangular pipe partially filled with charged liquid—for problem 2.36.

2.37 Write a program to generate associated Legendre polynomial, with $x = \cos\theta = 0.5$. You may use either series expansion or recurrence relations. Take $0 \leq n \leq 15, 0 \leq m \leq n$. Compare your results with those tabulated in standard tables.

2.38 The FORTRAN program of Fig. 2.16 uses the series expansion method to generate $j_n(x)$. Write a subroutine for generating $j_n(x)$ using recurrence relations. For $x = 2.0$ and $0 \leq n \leq 10$, compare your result with that obtained using the subroutine BESSEL of Fig. 2.16 and the values in standard tables. Which result do you consider to be more accurate? Explain.

2.39 Use the product generating function

$$G(x+y,t) = G(x,t)G(y,t)$$

to derive the *addition theorem*

$$J_n(x+y) = \sum_{m=-\infty}^{\infty} J_m(x) J_{n-m}(y)$$

Recall that

$$G(x,t) = \exp\left[\frac{x}{2}\left(t - \frac{1}{t}\right)\right] = \sum_{n=-\infty}^{\infty} t^n J_n(x)$$

2.40 Use the generating function to prove that:

$$\frac{1}{R} = \frac{1}{r_o}\sum_{n=0}^{\infty}(r/r_o)^n P_n(\cos\theta), \qquad r < r_o\ ,$$

$\dfrac{1}{R} = \dfrac{1}{r}\sum_{n=0}^{\infty}(r_o/r)^n P_n(\cos\theta)$, $r > r_o$, where $R = |\mathbf{r} - \mathbf{r}_o| = [r^2 - r_o^2 - 2rr_o\cos\alpha]^{1/2}$ and α is the angle between $\mathbf{r}$ and $\mathbf{r}_o$.

2.41 Show that

$$\int T_0(x)\,dx = T_1(x)$$
$$\int T_1(x)\,dx = \frac{1}{4}T_2(x) + \frac{1}{4}$$
$$\int T_n(x)\,dx = \frac{1}{2}\left(\frac{T_{n+1}(x)}{n+1} - \frac{T_{n-1}(x)}{n-1}\right), \qquad n > 1$$

so that integration can be done directly in Chebyshev polynomials.

2.42 A function is defined by

$$f(x) = \begin{cases} 1, & -1 \le x \le 1 \\ 0, & \text{otherwise} \end{cases}$$

(a) Expand $f(x)$ in a series of Hermite functions,

(b) expand $f(x)$ in a series of Laguerre functions.

2.43 By expressing E_θ^i and E_ϕ^i in terms of the spherical wave functions, show that Eq. (2.235) is valid.

2.44 By defining

$$\rho_n(x) = \frac{d}{dx} \ln \left[x h_n^{(2)}(x) \right], \qquad \sigma_n(x) = \frac{d}{dx} \ln \left[x j_n(x) \right] ,$$

show that the scattering amplitude coefficients can be written as

$$a_n = \frac{j_n(\alpha)}{h_n^{(2)}(\alpha)} \left[\frac{\sigma_n(\alpha) - m\sigma_n(m\alpha)}{\rho_n(\alpha) - m\sigma_n(m\alpha)} \right]$$
$$b_n = \frac{j_n(\alpha)}{h_n^{(2)}(\alpha)} \left[\frac{\sigma_n(m\alpha) - m\sigma_n(\alpha)}{\sigma_n(m\alpha) - m\rho_n(\alpha)} \right]$$

2.45 For the problem in Fig. 2.14, plot $|E_z^t|/|E_x^i|$ for $-a < z < a$ along the axis of the dielectric sphere of radius $a = 9$ cm in the $x - z$ plane. Take $E_o = 1$, $\omega = 2\pi \times 5 \times 10^9$ rad/s, $\epsilon_1 = 4\epsilon_o$, $\mu_1 = \mu_o$, $\sigma_1 = 0$. You may modify the program in Fig. 2.16 or write your own.

2.46 In analytical treatment of the radio-frequency radiation effect on the human body, the human skull is frequently modeled as a lossy sphere. Of major concern is the calculation of the normalized heating potential

$$\Phi(r) = \frac{1}{2}\sigma \frac{|E^t(r)|^2}{|E_o|^2} \qquad (\Omega \cdot m)^{-1} ,$$

where E^t is the internal electric field strength and E_o is the peak incident field strength. If the human skull can be represented by a homogeneous sphere of radius $a = 10$ cm, plot $\Phi(r)$ against the radial distance $-10 \le r = z \le 10$ cm. Assume an incident field as in Fig. 2.14 with $f = 1$ GHz, $\mu_r = 1$, $\epsilon_r = 60$, $\sigma = 0.9$ mhos/m, $E_o = 1$.

2.47 Instead of the homogeneous spherical model assumed in the previous problem, consider the multilayered spherical model shown in Fig. 2.25 with each region labeled by an integer p, such that $p = 1$ represents the central core region and $p = 4$ represents air. At $f = 2.45$ GHz, plot the heating potential along the x axis, y axis, and z axis. Assume the data given below.

Region p	Tissue	Radius (mm)	ϵ_r	σ (mho/m)
1	muscle	18.5	46	2.5
2	fat	19	6.95	0.29
3	skin	20	43	2.5
4	air		1	0
		$\mu_r = 1$		

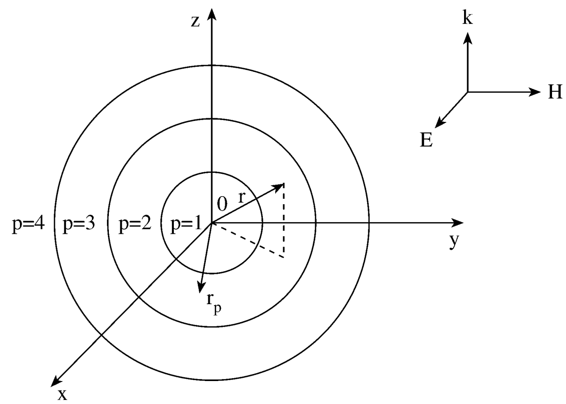


Figure 2.25
For problem 2.47, a multilayered spherical model of the human skull.

Note that for each region p, the resultant field consists of the transmitted and scattered fields and is in general given by

$$E_p(r, \theta, \phi) = E_o e^{j\omega t} \sum_{n=1}^{\infty} (-j)^n \frac{2n+1}{n(n+1)} \Big[a_{np} \mathbf{M}_{np}^{(4)}(k) + j b_{np} \mathbf{N}_{np}^{(4)}(k) + c_{np} \mathbf{M}_{np}^{(1)}(k_1) + j d_{np} \mathbf{N}_{np}^{(1)}(k1) \Big]$$

2.48 The absorption characteristic of biological bodies is determined in terms of the specific absorption rate (SAR) defined as the total power absorbed divided as the power incident on the geometrical cross section. For an incident power density of 1 mW/cm^2 in a spherical model of the human head,

$$\text{SAR} = 2\frac{Q_{\text{abs}}}{\pi a} \qquad \text{mW/cm}^3$$

where a is in centimeters. Using the above relation, plot SAR against frequency for $0.1 < f < 3$ GHz, $a = 10$ cm assuming frequency-dependent and dielectric properties of head as

$$\epsilon_r = 5\left(\frac{12 + (f/f_o)^2}{1 + (f/f_o)^2}\right)$$

$$\sigma = 6\left(\frac{1 + 62(f/f_o)^2}{1 + (f/f_o)^2}\right)$$

where f is in GHz and $f_o = 20$ GHz.

2.49 For the previous problem, repeat the calculations of SAR assuming a six-layered spherical model of the human skull (similar to that of Fig. 2.25) of outer radius

$a = 10$ cm. Plot P_a/P_i vs. frequency for $0.1 < f < 3$ GHz where

$$\frac{P_a}{P_i} = \frac{2}{\alpha^2} \sum (2n+1) \left[\mathrm{Re}\,(a_n + b_n) - \left(|a_n|^2 + |b_n|^2 \right) \right] ,$$

$P_a =$ absorbed power, $P_i =$ incident power, $\alpha = 2\pi a/\lambda$, λ is the wavelength in the external medium. Use the dimensions and electrical properties shown below.

Layer p	Tissue	Radius (mm)	ϵ_r	σ_o(mho/m)
1	brain	9	$5\nabla(f)$	$6\Delta(f)$
2	CSF	12	$7\nabla(f)$	$8\Delta(f)$
3	dura	13	$4\nabla(f)$	$8\Delta(f)$
4	bone	17.3	5	62
5	fat	18.5	6.95	0.29
6	skin	20	43	2.5

where $\mu_r = 1$,

$$\nabla(f) = \frac{1 + 12(f/f_o)^2}{1 + (f/f_o)^2} ,$$

$$\Delta(f) = \frac{1 + 62(f/f_o)^2}{1 + (f/f_o)^2} ,$$

f is in GHz, and $f_o = 20$ GHz. Compare your result with that from the previous problem.