An Anti-Diffusive Method for Simulating Interface Flows with a Five-Equation Model

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Numerical Methods for Multi-material Fluid Flows
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A Target (Open) Problem for Nuclear Safety Simulation

**Phenomenon**

- Liquid phase heated by a wall (pool boiling) at a given temperature $T_{\text{wall}}$.
- As $T_{\text{wall}}$ increases there is a transition from the Nucleate Boiling Regime to the Film Boiling Regime. This transition is still misunderstood.

![Nucleate Boiling](source: http://www.spaceflight.esa.int/users/fluids/TT_boiling.htm)

![Critical Flux](source: http://www.spaceflight.esa.int/users/fluids/TT_boiling.htm)

![Film Boiling](source: http://www.spaceflight.esa.int/users/fluids/TT_boiling.htm)

**A difficult problem**

- Determining the most significant physical scale is still an open issue
- Performing experiments is difficult.
Simulation Framework

Physical Framework

- Simulation of two-phase flows with interfaces
- Each fluid $k = 0, 1$ is compressible, EOS: $(\rho_k, \varepsilon_k) \rightarrow P_k$
- The interface is passively advected
- Model choice: Five-Equation Model (Allaire, Clerc, Kokh and also Murrone, Guillard, Massoni, Saurel, Périgaud, . . .)

Numerical Method

- Interface Capture Method: the interface is located by the mean of a color function $z$
- The variable $z$ is updated thanks to an advection PDE that is directly discretized
Issues & Questions

Drawback

The discretization for $z$ tends to smear the interface

Possible Cures

- Level-Set Type Method: conservativity issue (Osher, Sethian, Fedkiw, Smereka, Sussman, Gibou...)
- Anti-diffusive schemes (Després-Lagoutière, and also Reservoir Schemes: Alouges, Lorin, Le Coq, ...)

Is it possible to design an anti-diffusive solver for the Five-Equation system? Can we “extend” the Després-Lagoutière strategy to models different from its original framework?
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The Five-Equation Model with Isobaric Closure

Model Main Properties

\[ \begin{aligned}
\partial_t \rho u + \text{div} [\rho u \otimes u + P(z, y, \rho, \varepsilon) \text{Id}] &= 0 \\
\partial_t \rho e + \text{div} [(\rho e + P) u] &= 0 \\
\partial_t z + u \cdot \nabla z &= 0 \\
P_1(\rho_1, \varepsilon_1) &= P_0(\rho_0, \varepsilon_0) = P
\end{aligned} \]

\[ \begin{aligned}
\rho &= z \rho_1 + (1 - z) \rho_0, \quad \rho \varepsilon = z \rho_1 \varepsilon_1 + (1 - z) \rho_0 \varepsilon_0 \\
e &= \varepsilon + \frac{|u|^2}{2}, \\
y &= \frac{z \rho_1}{\rho}
\end{aligned} \]
Model Main Properties

- The system can be expressed in a fully conservative form
- Hyperbolicity is ensured when
  \[ \xi_k = \left( \frac{\partial \rho_k \varepsilon_k}{\partial P_k} \right)_{\rho_k} > 0 \quad \text{(reads } \gamma_k > 1 \text{ for perfect gases)} \]
- Eigenstructure of the system: \{u - c, u, u, u, u + c\}
- Sound velocity \( c \) given by
  \[ \xi c^2 = y \xi_1 c_1^2 + (1 - y) \xi_0 c_0^2, \]
  \[ \xi = z \xi_1 + (1 - z) \xi_0, \quad c_k = \text{pure fluid } k \text{ sound velocity} \]
- No entropy so far for the general case...
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The system reads (for regular solutions)

\[
\begin{align*}
D_t z &= 0, \\
D_t y &= 0 \\
\rho D_t \begin{bmatrix} 1/\rho \\ u \\ e \end{bmatrix} + \begin{bmatrix} -u \\ P \\ Pu \end{bmatrix}_x &= 0 \\
D_t \cdot &= \partial_t \cdot + u \partial_x \cdot.
\end{align*}
\]

Euler Coord.  \quad Lagrange Coord.

\[
\begin{align*}
a(x, t) &\leftrightarrow \tilde{a}(X, t) \\
\begin{cases}
\tilde{z}_t = 0, \\
\tilde{y}_t = 0
\end{cases} &\begin{cases}
\tilde{\rho} \\ \tilde{u} \\ \tilde{e} \end{cases}_t + \begin{bmatrix} -\tilde{u} \\ \tilde{P} \\ \tilde{P}\tilde{u} \end{bmatrix}_x = 0
\end{align*}
\]

Lagrange-Remap process

Map the Euler variable onto the Lagrange variable.

\[a^n \rightarrow \tilde{a}^n\]

Update the Lagrange variable by solving (L)

\[\tilde{a}^n \rightarrow \tilde{a}^{n+1}\]

Remap the Lagrange variable onto the Euler variable

\[\tilde{a}^{n+1} \rightarrow a^{n+1}\]
We only consider a single timestep: \( t^n \to t^{n+1} \).

\[
\begin{align*}
\text{Initial Lagrange & Euler Variable} & \quad a^n_j = \tilde{a}^n_j \\
\text{Updated Lagrange Variable} & \quad \tilde{a}^{n+1}_j \sim \tilde{a}_j \\
\text{Final Euler Variable} & \quad a^{n+1}_j
\end{align*}
\]
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Scheme for the Lagrange Step

Acoustic Scheme (Després)

\[
\begin{align*}
\left\{
\begin{array}{c}
\left( \frac{1}{\tilde{\rho}_j} - \frac{1}{\rho^n_j} \right) \\
\tilde{u}_j - u^n_j \\
\tilde{e}_j - e^n_j
\end{array}
\right) + \frac{\lambda}{\rho^n_j} \left(
\begin{array}{c}
-u_{j+1/2} + u_{j-1/2} \\
-P_{j+1/2} - P_{j-1/2} \\
P_{j+1/2}u_{j+1/2} - P_{j-1/2}u_{j-1/2}
\end{array}
\right) = 0 \\
\tilde{z}_j = z^n_j, \quad \tilde{y}_j = y^n_j, \quad \lambda = \Delta t / \Delta x
\end{align*}
\]

Flux Formulas

\[
\begin{align*}
(p\c)^*_{j+1/2} &= \sqrt{\max[\rho^n_j(c^n_j)^2, \rho^n_{j+1}(c^n_{j+1})^2]} \min(\rho^n_j, \rho^n_{j+1}) \\
P_{j+1/2} &= \frac{1}{2} (P^n_j + P^n_{j+1}) - \frac{1}{2} (p\c)^*_{j+1/2} (u^n_{j+1} - u^n_j) \\
\left( u_{j+1/2} \right) &= \frac{1}{2} (u^n_j + u^n_{j+1}) - \frac{1}{2} \frac{1}{(p\c)^*_{j+1/2}} (P^n_{j+1} - P^n_j)
\end{align*}
\]

This step preserves \((P, u)\)-constant profiles.
Scheme for the Remapping Step

General Form

\[ \rho W = (\rho y, \rho, \rho u, \rho e)^T \]

\[ \rho_j^{n+1} W_j^{n+1} - \rho_j \tilde{W}_j + \lambda \left( u_{j+1/2} \tilde{\rho}_{j+1/2} \tilde{W}_{j+1/2} - u_{j-1/2} \tilde{\rho}_{j-1/2} \tilde{W}_{j-1/2} \right) = 0 \]

\[ (z_j^{n+1} - z_j^n) + \lambda \left( u_{j+1/2} \tilde{z}_{j+1/2} - u_{j-1/2} \tilde{z}_{j-1/2} \right) - \lambda z_j^n (u_{j+1/2} - u_{j-1/2}) = 0 \]

Building the scheme boils down to specify the following terms

\[ \tilde{y}_{j+1/2}, \quad \tilde{\rho}_{j+1/2}, \quad \tilde{u}_{j+1/2}, \quad \tilde{\varepsilon}_{j+1/2}, \quad \tilde{z}_{j+1/2} \]
Defining the Fluxes

\[ \tilde{z}_{j+1/2} = ? \]

\[ \tilde{u}_{j+1/2} = ? \]

\[ \tilde{y}_{j+1/2} = ? \]

\[ \tilde{\rho}_{j+1/2} = ? \]

\[ (\tilde{\rho\varepsilon})_{j+1/2} = ? \]

Enforce consistency for \( \tilde{y}_{j+1/2}, \tilde{\rho}_{j+1/2}, \tilde{\varepsilon}_{j+1/2} \).

Upwind choice according to the sign of \( u_{j+1/2} \) for phasic quantities \( \rho_0, \rho_1 \) and \( \rho_0 \varepsilon_0, \rho_1 \varepsilon_1 \).

Upwind choice too for \( u \).
Defining the Fluxes

\[ \tilde{z}_{j+1/2} = ? \]

\[ \tilde{u}_{j+1/2} = ? \]

\[ \tilde{y}_{j+1/2} = \tilde{z}_{j+1/2}(\rho_1)_{j+1/2}/\tilde{\rho}_{j+1/2} \]

\[ \tilde{\rho}_{j+1/2} = \tilde{z}_{j+1/2}(\rho_1)_{j+1/2} + (1-\tilde{z}_{j+1/2})(\rho_0)_{j+1/2} \]

\[ (\rho\varepsilon)_{j+1/2} = \tilde{z}_{j+1/2}(\rho_1\varepsilon_1)_{j+1/2} + (1-\tilde{z}_{j+1/2})(\rho_0\varepsilon_0)_{j+1/2} \]

Enforce consistency for \( \tilde{y}_{j+1/2}, \tilde{\rho}_{j+1/2}, \tilde{\varepsilon}_{j+1/2} \).

Upwind choice \((j + 1/2 = \text{upw})\) according to the sign of \( u_{j+1/2} \) for phasic quantities \( \rho_0, \rho_1 \) and \( \rho_0\varepsilon_0, \rho_1\varepsilon_1 \).

Upwind choice too for \( u \).
Defining the Fluxes

\[ \tilde{z}_{j+1/2} = ? \]

\[ \tilde{u}_{j+1/2} = ? \]

\[ \tilde{y}_{j+1/2} = \tilde{z}_{j+1/2} (\rho_1)_{\text{upw}} / \tilde{\rho}_{j+1/2} \]

\[ \tilde{\rho}_{j+1/2} = \tilde{z}_{j+1/2} (\rho_1)_{\text{upw}} + (1 - \tilde{z}_{j+1/2}) (\rho_0)_{\text{upw}} \]

\[ (\tilde{\rho\varepsilon})_{j+1/2} = \tilde{z}_{j+1/2} (\rho_1 \varepsilon_1)_{\text{upw}} + (1 - \tilde{z}_{j+1/2}) (\rho_0 \varepsilon_0)_{\text{upw}} \]

Enforce consistency for \( \tilde{y}_{j+1/2}, \tilde{\rho}_{j+1/2}, \tilde{\varepsilon}_{j+1/2} \).

Upwind choice \((j + 1/2 = \text{upw})\) according to the sign of \( u_{j+1/2} \) for phasic quantities \( \rho_0, \rho_1 \) and \( \rho_0 \varepsilon_0, \rho_1 \varepsilon_1 \).

Upwind choice too for \( u \).
Defining the Fluxes

\[ \tilde{z}_{j+1/2} =? \]

\[ \tilde{u}_{j+1/2} = u_{\text{upw}} \]

\[ \tilde{\eta}_{j+1/2} = \tilde{z}_{j+1/2}(\rho_1)_{\text{upw}}/\tilde{\rho}_{j+1/2} \]

\[ \tilde{\rho}_{j+1/2} = \tilde{z}_{j+1/2}(\rho_1)_{\text{upw}} + (1 - \tilde{z}_{j+1/2})(\rho_0)_{\text{upw}} \]

\[ (\rho\varepsilon)_{j+1/2} = \tilde{z}_{j+1/2}(\rho_1\varepsilon_1)_{\text{upw}} + (1-\tilde{z}_{j+1/2})(\rho_0\varepsilon_0)_{\text{upw}} \]

Enforce consistency for \( \tilde{\eta}_{j+1/2}, \tilde{\rho}_{j+1/2}, \tilde{\varepsilon}_{j+1/2} \).

Upwind choice (\( j + 1/2 = \text{upw} \)) according to the sign of \( u_{j+1/2} \) for phasic quantities \( \rho_0, \rho_1 \) and \( \rho_0\varepsilon_0, \rho_1\varepsilon_1 \).

Upwind choice too for \( u \).
Defining the Fluxes

\[ \tilde{z}_{j+1/2} = ? \]

\[ \tilde{u}_{j+1/2} = u_{\text{upw}} \]

\[ \tilde{y}_{j+1/2} = \tilde{z}_{j+1/2} (\rho_1)_{\text{upw}} / \tilde{\rho}_{j+1/2} \]

\[ \tilde{\rho}_{j+1/2} = \tilde{z}_{j+1/2} (\rho_1)_{\text{upw}} + (1 - \tilde{z}_{j+1/2}) (\rho_0)_{\text{upw}} \]

\[ (\rho \varepsilon)_{j+1/2} = \tilde{z}_{j+1/2} (\rho_1 \varepsilon_1)_{\text{upw}} + (1 - \tilde{z}_{j+1/2}) (\rho_0 \varepsilon_0)_{\text{upw}} \]

Enforce consistency for \( \tilde{y}_{j+1/2}, \tilde{\rho}_{j+1/2}, \tilde{\varepsilon}_{j+1/2} \).

Upwind choice \((j + 1/2 = \text{upw})\) according to the sign of \( u_{j+1/2} \) for phasic quantities \( \rho_0, \rho_1 \) and \( \rho_0 \varepsilon_0, \rho_1 \varepsilon_1 \).

Upwind choice too for \( u \).
Consistency for the Flux $\tilde{z}_{j+1/2}$

\[
\min(z_j^n, z_{j+1}^n) \leq \tilde{z}_{j+1/2} \leq \max(z_j^n, z_{j+1}^n) \implies \text{consistency}
\]
Consistency for the Flux $\tilde{z}_{j+1/2}$

Consistency "Trust Interval"

$$\tilde{z}_{j+1/2} \in [m_{j+1/2}, M_{j+1/2}]$$

$$m_{j+1/2} = \min(z_j^n, z_{j+1}^n), \quad M_{j+1/2} = \max(z_j^n, z_{j+1}^n)$$
Ad hoc properties provide a simple stability criterion.

Suppose \( u_{j+1/2} > 0 \)
Examine the \( j^{\text{th}} \) cell if \( u_{j-1/2} > 0 \)
Ad hoc properties provide a simple stability criterion.

Suppose $u_{j+1/2} > 0$
Examine the $j^{th}$ cell if $u_{j-1/2} > 0$

Stability Criterion

$z_j^{n+1} \in [m_{j-1/2}, M_{j-1/2}] \implies \text{Stability}$
Ad hoc properties provide a simple stability criterion. Suppose $u_{j+1/2} > 0$.
Examine the $j^{th}$ cell if $u_{j-1/2} > 0$.

Stability Criterion

$$\tilde{z}_{j+1/2} ? \implies z_j^{n+1} \in [m_{j-1/2}, M_{j-1/2}] \implies \text{Stability}$$
Sufficient Stability Condition (case $u_{j-1/2} > 0$ and $u_{j+1/2} > 0$)

$$
\tilde{z}_{j+1/2}^n \quad ?
\
\downarrow
\
\uparrow
\
\tilde{z}_{j+1/2}^n \in [m_{j-1/2}, M_{j-1/2}] \Longrightarrow \text{Stability for } z
$$
The Numerical Remap Method

The Numerical Remap Method

\[ a_j = z_j^n + \left( \frac{u_{j-1/2}}{u_{j+1/2}} - \frac{1}{\lambda u_{j+1/2}} \right) (m_{j-1/2} - z_j^n) \]
\[ A_j = z_j^n + \left( \frac{u_{j-1/2}}{u_{j+1/2}} - \frac{1}{\lambda u_{j+1/2}} \right) (M_{j-1/2} - z_j^n) \]
\[ \tilde{z}_{j+1/2} \in [a_j, A_j] \]

Sufficient Stability Condition (case \( u_{j-1/2} > 0 \) and \( u_{j+1/2} > 0 \))

\[ z_j^{n+1} \in [m_{j-1/2}, M_{j-1/2}] \rightarrow \text{Stability for } z \]
Stability and Consistency Trust Interval for $z$

$$\tilde{z}_{j+1/2} \in [a_j, A_j] \cap [m_{j+1/2}, M_{j+1/2}]$$
Suppose \( u_{j-1/2} > 0 \) and \( u_{j+1/2} > 0 \).

**Similar Lines**

**Stability for \( y \)**

\[
\begin{align*}
y_{j+1}^{n+1} &\in [m_{j-1/2}^y, M_{j-1/2}^y] \\
\tilde{y}_{j+1/2} &= \frac{(\tilde{\rho}_0)_j \tilde{z}_{j+1/2}}{\tilde{\rho}_{j+1/2}} \in [d_j, D_j] \\
\tilde{z}_{j+1/2} &\in [b_j, B_j]
\end{align*}
\]

**Stability for \( y^n \)**

\[
\begin{align*}
m_{j-1/2}^y &= \min(y_{n-1}^n, y_j^n) \\
M_{j-1/2}^y &= \max(y_{n-1}^n, y_j^n) \\
d_j &= M_{j-1/2}^y + \frac{\rho_j^n}{\lambda u_{j+1/2} \tilde{\rho}_{j+1/2}} (y_j^n - M_{j-1/2}^y) \\
D_j &= m_{j-1/2}^y + \frac{\rho_j^n}{\lambda u_{j+1/2} \tilde{\rho}_{j+1/2}} (y_j^n - m_{j-1/2}^y) \\
b_j &= z_j^n + \frac{\rho_j^n (y_j^n - M_{j-1/2}^y)}{(\rho_1)_j (1 - M_{j+1/2}^y) + (\rho_2)_j M_{j+1/2}^y} \\
B_j &= z_j^n + \frac{\rho_j^n (y_j^n - m_{j-1/2}^y)}{(\rho_1)_j^n (1 - m_{j+1/2}^y) + (\rho_2)_j^n m_{j+1/2}^y}
\end{align*}
\]
Suppose $u_{j-1/2} > 0$ and $u_{j+1/2} > 0$.

\[
\begin{align*}
\tilde{y}_{j+1/2} &= \frac{(\tilde{\rho}_0)_j \tilde{z}_{j+1/2}}{\tilde{\rho}_{j+1/2}} \in \left[ d_j, D_j \right] \\
\tilde{z}_{j+1/2} &\in \left[ b_j, B_j \right]
\end{align*}
\]

\[
\left\{ \begin{array}{c}
m_{j-1/2}^y = \min(y^n_{j-1}, y^n_j) \\
M_{j-1/2}^y = \max(y^n_{j-1}, y^n_j)
\end{array} \right.
\]

explicit expression for $d_j$, $D_j$, $b_j$, $B_j$. 

Similar Lines

Stability for the Mass Fraction $y$

\[
y_{j+1}^{n+1} \in \left[ m_{j-1/2}^y, M_{j-1/2}^y \right]
\]
Constraints for the flux $\tilde{z}_{j+1/2}$

Suppose $u_{j-1/2} > 0$ and $u_{j+1/2} > 0$.

We have a “trust interval” for $\tilde{z}_{j+1/2}$ that ensures stability for both $y$ and $z$, and consistency for the flux $\tilde{z}_{j+1/2}$.

$$\tilde{z}_{j+1/2} \in \left[ m_{j-1/2}, M_{j-1/2} \right] \cap \left[ a_j, A_j \right] \cap \left[ b_j, B_j \right]$$
Constraints for the flux $\tilde{z}_{j+1/2}$

Suppose $u_{j-1/2} > 0$ and $u_{j+1/2} > 0$.

We have a “trust interval” for $\tilde{z}_{j+1/2}$ that ensures stability for both $y$ and $z$, and consistency for the flux $\tilde{z}_{j+1/2}$.

$$\tilde{z}_{j+1/2} \in \left[ m_{j-1/2}, M_{j-1/2} \right] \cap \left[ a_j, A_j \right] \cap \left[ b_j, B_j \right] \neq \emptyset$$
Constraints for the flux $\tilde{z}_{j+1/2}$

Suppose $u_{j-1/2} > 0$ and $u_{j+1/2} > 0$.

We have a “trust interval” for $\tilde{z}_{j+1/2}$ that ensures stability for both $y$ and $z$, and consistency for the flux $\tilde{z}_{j+1/2}$.

\[
\tilde{z}_{j+1/2} \in \left[m_{j-1/2}, M_{j-1/2}\right] \cap \left[a_j, A_j\right] \cap \left[b_j, B_j\right] \neq \emptyset
\]

$z^n_j$
Suppose \( u_{j-1/2} > 0 \) and \( u_{j+1/2} > 0 \).

We have a “trust interval” for \( \tilde{z}_{j+1/2} \) that ensures stability for both \( y \) and \( z \), and consistency for the flux \( \tilde{z}_{j+1/2} \).

\[
\tilde{z}_{j+1/2} \in \left[ m_{j-1/2}, M_{j-1/2} \right] \cap \left[ a_j, A_j \right] \cap \left[ b_j, B_j \right] \neq \emptyset
\]

\( 
\bigcup
\]

\( \tilde{z}_j^n \)

How should we choose \( \tilde{z}_{j+1/2} \)?
Choosing the $z$-flux $\tilde{z}_{j+1/2}$ (1)

Suppose $u_{j-1/2} > 0$ and $u_{j+1/2} > 0$.

Després-Lagoutièrie strategy

We choose the most downwind possible value for $\tilde{z}_{j+1/2}$ such that

$$\tilde{z}_{j+1/2} \in \left[ m_{j-1/2}, M_{j-1/2} \right] \cap \left[ a_j, A_j \right] \cap \left[ b_j, B_j \right]$$

Choosing the $\tilde{z}_{j+1/2}$ value is also an explicit step: no CPU cost is needed, nor any recursive procedure.
Choosing the flux $\tilde{z}_{j+1/2}$ (2)

What if $u_{j-1/2} < 0$ and $u_{j+1/2} > 0$?

For sake of “security” we opt for the upwind choice, i.e.

$$\tilde{z}_{j+1/2} = z_j^n$$

Other cases

The cases $u_{j-1/2} > 0$, $u_{j+1/2} < 0$ and $u_{j-1/2} < 0$, $u_{j+1/2} < 0$ can be examined following the same lines and provide similar formulas for the flux $\tilde{z}_{j+1/2}$. 
The overall scheme reads

\[
\begin{aligned}
\rho_j^{n+1} W_j^{n+1} - \rho_j^n W_j^n + \lambda (F_{j+1/2} - F_{j-1/2}) &= 0 \\
(z_j^{n+1} - z_j^n) + \lambda (u_{j+1/2} \tilde{z}_{j+1/2} - u_{j-1/2} \tilde{z}_{j-1/2}) - \lambda z_j^n (u_{j+1/2} - u_{j-1/2}) &= 0
\end{aligned}
\]

\[
\rho W = (\rho y, \rho, \rho u, \rho e)^T \\
F_{j+1/2} = \tilde{\rho}_{j+1/2} \tilde{W}_{j+1/2} u_{j+1/2} + (0, 0, P_{j+1/2}, P_{j+1/2} u_{j+1/2})^T
\]
Introduction

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iso-($P, U$) Profiles

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iso-\((P, U)\) Profiles

The scheme inherits the properties of the original Five-Equation and solver.

If \((\rho W, z)^n\) is such that

1. \(\forall j, \ P_j^n = P\) and \(u_j^n = u\)
2. If \(z\) varies across the cell interface \(j + 1/2\) then
   \[
   (\rho_k)_j^n = (\rho_k)_{j+1}^n, \ k = 0, 1
   \]

Then \(\forall j, \ P_j^{n+1} = P\) and \(u_j^{n+1} = u\)

Remark

Assumption (2) is not necessary for some EOS (Perfect Gases, Stiffened Gas)
Interface Advection (1)

Test Description
Advection of a 1D “bubble” (pulse) involving two perfect gases

Inner bubble state
\[ \gamma = 1.4, \quad \rho = 50.0 \]
\[ P = 10^5, \quad u = 1.0 \]

Outer bubble State
\[ \gamma = 4.4, \quad \rho = 100 \]
\[ P = 10^5, \quad u = 1.0 \]

- The domain is discretized over a 100 cells mesh
- Periodic boundary conditions
- \( t = 1000 \) s (1000 turns)
Interface Advection (2): Initial State
Interface Advection (3): Color Function at $t = 2 \text{ s}$
Interface Advection (4): Color Function at $t = 20 \text{s}$
Interface Advection (5): Color Function at $t = 1000$ s
Interface Advection (6): Pressure & Velocity at $t = 1000$ s
Interface Advection (7): Numerical Diffusion of the Color Function

![Graph showing numerical diffusion over time steps for upwind and downwind conditions. The graph plots the percent of cell number with numerical diffusion against time steps. The x-axis represents time steps ranging from 0 to 500, while the y-axis represents the percent of cell number with numerical diffusion, ranging from 0 to 100. The graph shows two lines: one for upwind (blue dashed line) and one for downwind (red line).]
## Shock Tube 1 (1)

### Test Description

Riemann Problem with two perfect gases

<table>
<thead>
<tr>
<th>Left State</th>
<th>Right State</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\gamma = 1.4$</td>
<td>$\gamma = 2.4$</td>
</tr>
<tr>
<td>$\rho = 1.0$</td>
<td>$\rho = 0.125$</td>
</tr>
<tr>
<td>$P = 1.0$</td>
<td>$P = 0.1$</td>
</tr>
<tr>
<td>$u = 0.0$</td>
<td>$u = 0.0$</td>
</tr>
</tbody>
</table>

- The domain is discretized over a 300 cells mesh.
- $t = 0.14s$
Shock Tube 1 (2): Velocity

The graph shows the velocity profiles for Shock Tube 1 with different advection methods: upwind, downwind, and exact. The graph compares the numerical results with the exact solution.
Shock Tube 1 (3): Pressure

![Pressure Profile](image)

Shock Tube 1

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Anti-Diffusive Meth. for Interface Flows with a 5-Eq. Model
Shock Tube 1 (4): Color Function

![Color Function Graph]

- **Shock Tube 1**
- **Shock Tube 2**
- Advection of a 2D Interface
- 2D Shock/Interface Interaction
- Kelvin-Helmholtz Instability
- Gas Bulk Rising Towards a Free Surface

**Anti-Diffusive Meth. for Interface Flows with a 5-Eq. Model**
Shock Tube 1 (5): Mass Fraction
Shock Tube 1 (6): Density
Shock Tube 1 (7): Numerical Diffusion of The Color Function

![Graph showing numerical diffusion of color function over time steps. The graph displays two lines: one dotted line representing upwind diffusion and one solid line representing downwind diffusion. The x-axis represents time steps, and the y-axis represents the percent of cell number with numerical diffusion. The data points for both lines are plotted at various time steps, showing an increase in the percent of cell number with numerical diffusion as time progresses.]
Shock Tube 1 (9): Pressure (50000 cells)

The graph shows the pressure profile downwind for Shock Tube 1. The data is compared to the exact solution. The plot indicates a sharp decrease in pressure at the shock front, followed by a plateau in the post-shock region.
Shock Tube 1 (10): Color Function (50000 cells)
Shock Tube 1 (11): Mass Fraction (50000 cells)
Shock Tube 1 (12): Density (50000 cells)
Shock Tube 2 (1)

Test Description
Riemann Problem with two perfect gases

Left State
\[ \gamma = 1.67 \]
\[ \rho = 14.549030 \]
\[ P = 194.0 \times 10^5 \]
\[ u = 0.0 \]

Right State
\[ \gamma = 1.40 \]
\[ \rho = 1.163550 \]
\[ P = 10^5 \]
\[ u = 0.0 \]

- The domain is discretized over a 500 cells mesh.
- \[ t = 2.4 \times 10^{-4} \text{ s} \]
Shock Tube 2 (2): Velocity

Anti-Diffusive Meth. for Interface Flows with a 5-Eq. Model
Shock Tube 2 (3): Pressure
Shock Tube 2 (4): Color Function
Shock Tube 2 (5): Mass Fraction

![Graph showing mass fraction over time for Shock Tube 2 with upwind and downwind markers.](image)
Shock Tube 2 (6): Density

![Graph showing density profiles for upwind and downwind conditions.]

- **Upwind** (black line)
- **Downwind** (red line)

The graph illustrates the density distribution for shock tubes 1 and 2, showing the advection of a 2D interface and 2D shock/interface interaction, as well as Kelvin-Helmholtz instability and gas bulk rising towards a free surface.
Shock Tube 2 (7): Numerical Diffusion of The Color Function

![Graph showing numerical diffusion of color function over time steps. The graph plots the percentage of cell number with numerical diffusion against time steps. The data points are connected with dotted lines, one for upwind and one for downwind. The upwind line shows a steeper increase in diffusion compared to the downwind line.]
Advection of a 2D Interface (1)

Test Description

- advect a 2D interface
  - Domain discretized over a $100 \times 100$ mesh
  - Velocity $(u_x, u_y) = (1.0, 1.0)$

Color Function ($t = 0$)
Advection of a 2D Interface (2)

Color Function at $t = 0.2$ s

upwind  downwind
Advection of a 2D Interface (3)

Color Function at $t = 5 \text{ s}$

upwind

downwind
Advection of a 2D Interface (4): Numerical Diffusion of the Color Function
2D Shock/Interface Interaction (1)

Test Description

A planar shock hits a bubble initially at rest.

- Domain discretized over a $100 \times 100$ mesh
- Two perfect gases
2D Shock/Interface Interaction (2)

Color Function at $t = 0.7$ s

upwind

downwind
2D Shock/Interface Interaction (3)

Mass Fraction at $t = 0.7\, s$

upwind

downwind
2D Shock/Interface Interaction (4)

Density at $t = 0.7$ s

upwind

downwind

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Anti-Diffusive Meth. for Interface Flows with a 5-Eq. Model
2D Shock/Interface Interaction (5): Numerical Diffusion of the Color Function

The graph illustrates the percentage of cell number with numerical diffusion over time steps. The blue line represents the upwind case, while the red line represents the downwind case. The x-axis represents time steps, and the y-axis represents the percentage of cell number with numerical diffusion.
Kelvin-Helmholtz Instability (1)

The domain is discretized over a $1000 \times 1000$ mesh.
Kelvin-Helmholtz Instability (2)

**upwind**

**downwind**
Kelvin-Helmholtz Instability (3)

upwind

downwind
Kelvin-Helmholtz Instability (4)

upwind

downwind
Kelvin-Helmholtz Instability (5)

upwind

downwind

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Anti-Diffusive Meth. for Interface Flows with a 5-Eq. Model
Kelvin-Helmholtz Instability (6)

upwind

downwind
Kelvin-Helmholtz Instability (7)

upwind

downwind
Kelvin-Helmholtz Instability (9): Numerical Diffusion
Gas Bulk Rising Towards a Free Surface (1)

Joint work with O. Grégoire (CEA) & P. Salvatore (CEA).

Test Description

A 1 m$^3$ volume of gas (Perfect Gas) initially located near the bottom of a 10 m high water column (Stiffened Gas) starts rising towards the surface.

- The domain discretized over a 100 $\times$ 400 mesh
- Density ratio is 500
- Surface Tension via CSF Method (Brackbill)
- MUSCL reconstruction for the interface states
Gas Bulk Rising Towards a Free Surface (2)

Color Function

$t = 0 \text{ s}$

$t = 0.5 \text{ s}$
Gas Bulk Rising Towards a Free Surface (3)

Color Function

$t = 1.04 \text{ s}$

$t = 1.37 \text{ s}$
Gas Bulk Rising Towards a Free Surface (4)

Color Function

$t = 2.78 \text{ s}$
Gas Bulk Rising Towards a Free Surface (5)

Density

\[ t = 0.96 \, \text{s} \]

\[ t = 2.62 \, \text{s} \]
Conclusion & Perspectives

- Design of a Lagrange-Remap scheme for the Five-Equation system with isobaric closure
- The scheme is conservative for $\rho y$, $\rho$, $\rho u$, $\rho e$
- Good treatment of the Riemann Invariants across the material interface
- The scheme works “out of the box”: no specific “numerical tuning” is required to optimize the scheme performance
- Anti-diffusive interface advection for both mass fraction $y$ & color function $z$
- “Positivity” for both $y$ & $z$ variables
- no extra CPU cost
- The Després-Lagoutière approach is not restricted to its originating framework

Application to other models (see Dellacherie, Jaouen)?