Comoving-frame and Laboratory-frame Nonequilibrium Grey Radiation Diffusion Approximations in the Non-Relativistic Limit

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Motivation

- The purpose of this work is to contrast the non-equilibrium grey diffusion approximations in the Eulerian and comoving frames in the limit of non-relativistic material flow.
- We define the non-relativistic regime as $v/c < .01$.
- It has been suggested to us that any diffusion approximation in the Eulerian frame must be inherently flawed because “there is no diffusion in the Eulerian frame”.
- Our results indicate that this is not the case in the nonrelativistic limit.
- Indeed, the Eulerian-frame equation further justifies the approximate manner in which the comoving-frame diffusion approximation is usually used in the nonrelativistic limit.

Overview

- The fluid equations with grey radiation coupling to O(v/c).
- The Eulerian-frame radiation energy and momentum equations, the P$_1$ equations, and the diffusion equation, to O(v/c).
- The comoving-frame radiation energy and momentum equations, the P$_1$ equations, and the diffusion equation, to O(v/c).
- The asymptotic equilibrium-diffusion limit.
- A simplified Eulerian-frame equation.
- Summary of results.

Fluid Equations with Radiation Coupling

- Conservation of fluid mass:

$$\partial_t \rho + \partial_i \left(\rho v_i\right) = 0 \,, \tag{1}$$

- Conservation of fluid momentum:

$$\partial_t \left(\rho v_i\right) + \partial_j \left(\rho v_i v_j\right) + \partial_i p = \frac{\sigma_t}{c} F_{0,i} - \frac{v_i}{c} \sigma_a \left(a T^4 - E_0\right) \,, \tag{2}$$

- Conservation of total fluid energy:

$$\partial_t \left(\frac{1}{2}\rho v^2 + \rho e\right) + \partial_i \left[\left(\frac{1}{2}\rho v^2 + \rho e + p\right) v_i\right] =$$

$$-c\sigma_a \left(a T^4 - E_0\right) + \frac{\sigma_t}{c} v_i F_{0,i} \,, \tag{3}$$

- where the radiation coupling is most naturally given in terms of comoving-frame quantities.

Fluid Equations with Radiation Coupling

- The Eulerian-to-comoving transformations to O(v/c) are given by:

$$E_0 = E - \frac{2}{c^2} v_i \left(F_i - v_i E - v_j P_{ij} \right) \; , \tag{4}$$

$$F_{0,i} = F_i - v_i E - v_j P_{ij} \; , \tag{5}$$

$$P_{0,i,j} = P_{i,j} - \frac{1}{c^2} \left(v_i F_j + F_i v_j \right) \; . \tag{6}$$

- These relationships can be used to express the fluid-radiation coupling in terms of Eulerian-frame quantities.

The Eulerian-Frame Radiation Equations

- Conservation of radiation momentum:

$$\frac{1}{c^2}\partial_t F_i + \partial_j P_{ij} = -\frac{\sigma_t}{c} F_{0,i} + \frac{v_i}{c}\sigma_a \left(aT^4 - E_0\right) , \qquad (7)$$

- Conservation of radiation energy:

$$\partial_t E + \partial_i F_i = c\sigma_a \left(aT^4 - E_0\right) - \frac{\sigma_t}{c} v_i F_{0,i} , \qquad (8)$$

- To obtain the P_1 equations, we need only set $P_{ij} = \frac{1}{3}\delta_{ij} E$ in the radiation momentum equation.

The Eulerian-Frame P_1 Equations

- Conservation of radiation momentum:

$$\frac{1}{c^2}\partial_t F_i + \frac{1}{3}\partial_i E = -\frac{\sigma_t}{c}F_{0,i} + \frac{v_i}{c}\sigma_a\left(aT^4 - E_0\right) , \qquad (9)$$

- Conservation of radiation energy:

$$\partial_t E + \partial_i F_i = c\sigma_a\left(aT^4 - E_0\right) - \frac{\sigma_t}{c}v_i F_{0,i} , \qquad (10)$$

where

$$F_{0,i} = F_i - v_i\frac{4}{3}E . \qquad (11)$$

- To obtain the diffusion approximation, we need *only* set the time-derivative of the flux to zero in the radiation momentum equation.

The Eulerian-Frame Diffusion Equation

- This causes momentum conservation to be lost, but energy is still conserved.
- Solving for the flux, we get

$$F_i = -\frac{c}{3\sigma_t}\partial_i E + v_i \frac{4}{3} E + v_i \frac{\sigma_a}{\sigma_t}\left(aT^4 - E_0\right) . \qquad (12)$$

- Substituting from Eq.(12) into Eq.(10), we obtain the Eulerian-frame grey diffusion equation:

$$\partial_t E - \partial_i \frac{c}{3\sigma_t}\partial_i E + \partial_i \left\{ v_i \left[\frac{4}{3} E + \frac{\sigma_a}{\sigma_t}\left(aT^4 - E_0\right)\right]\right\} =$$

$$c\sigma_a \left(aT^4 - E_0\right) - \frac{\sigma_t}{c} v_i F_{0,i} . \qquad (13)$$

The Comoving-Frame Radiation Equations

- The Radiation Momentum Equation:

$$\frac{1}{c^2}\partial_t F_{0,i} + \frac{1}{c^2} v_j \partial_t P_{0,ij} + \partial_j P_{0,ij} +$$

$$\frac{1}{c^2} F_{0,j} \partial_j v_i + \frac{1}{c^2}\partial_j \left(F_{0,i} v_j \right) = -\frac{\sigma_t}{c} F_{0,i} \,. \qquad (14)$$

- The Radiation Energy Equation:

$$\partial_t E_0 + \frac{1}{c^2} v_i \partial_t F_{0,i} + \partial_i F_{0,i} + \partial_i \left(E_0 v_i \right) +$$

$$\partial_i \left(P_{0,ij} v_j \right) - v_i \partial_j P_{0,ij} = c\sigma_a \left(aT^4 - E_0 \right) \,. \qquad (15)$$

- To obtain the comoving-frame P_1 equations, we need simply assume that

$$P_{0,ij} = \frac{1}{3}\delta_{ij} E_0 \,. \qquad (16)$$

The Comoving-Frame P_1 Equations

- The Radiation Momentum Equation:

$$\frac{1}{c^2}\partial_t F_{0,i} + \frac{1}{3c^2} v_i \partial_t E_0 + \frac{1}{3}\partial_i E_0 +$$

$$\frac{1}{c^2} F_{0,j}\partial_j v_i + \frac{1}{c^2}\partial_j \left(F_{0,i} v_j\right) = -\frac{\sigma_t}{c} F_{0,i}\,. \qquad (17)$$

- The Radiation Energy Equation:

$$\partial_t E_0 + \frac{1}{c^2} v_i \partial_t F_{0,i} + \partial_i F_{0,i} + \partial_i \left(v_i E_0\right) +$$

$$\frac{1}{3} E_0 \partial_i v_i = c\sigma_a \left(aT^4 - E_0\right)\,. \qquad (18)$$

- Since the comoving frame is not inertial, we must transform these equations to the Eulerian frame to determine if they are conservative.

The Comoving-Frame P_1 Equations

- Making this transformation, we obtain the following momentum and energy equations:

$$\frac{1}{c^2}\partial_t F_i + \partial_j P^e_{ij} = -\frac{\sigma_t}{c} F_{0,i} + \frac{v_i}{c}\sigma_a \left(aT^4 - E_0\right) \ , \qquad (19)$$

where

$$P^e_{ij} = \frac{1}{3} E\delta_{ij} - \frac{1}{c^2}\left[\frac{2}{3}\left(v_k F_k\right)\delta_{ij} - v_i F_j - F_i v_j\right] . \qquad (20)$$

$$\partial_t E + \partial_i F_i = c\sigma_a \left(aT^4 - E_0\right) - \frac{\sigma_t}{c} v_i F_{0,i} \, . \qquad (21)$$

- These equations are conservative, and the energy equation is actually identical to that of the Eulerian-frame P_1 equation.

The Comoving-Frame Diffusion Approximation

- To obtain the grey diffusion approximation, we must set all velocity dependent terms to zero in Eq.(17), and we must set the time derivative of the flux to zero in *both* Eqs. (17) and (18).

$$\frac{1}{3}\partial_i E_0 = -\frac{\sigma_t}{c} F_{0,i} \,, \tag{22}$$

and

$$\partial_t E_0 + \partial_i F_{0,i} + \partial_i \left(v_i E_0\right) + \frac{1}{3} E_0 \partial_i v_i = c\sigma_a \left(aT^4 - E_0\right) \,. \tag{23}$$

- Solving Eq.(22) for the flux, we get Fick's law:

$$F_{0,i} = -\frac{c}{3\sigma_t}\partial_i E_0 \,. \tag{24}$$

The Comoving-Frame Diffusion Approximation

- Substituting from Eq.(24) into Eq.(23), we get the following diffusion equation for the radiation energy density:

$$\partial_t E_0 - \partial_i \left(\frac{c}{3\sigma_t} \partial_i E_0 \right) + \partial_i \left(v_i E_0 \right) +$$

$$\frac{1}{3} E_0 \partial_i v_i = c\sigma_a \left(aT^4 - E_0 \right) \ . \qquad (25)$$

- To investigate conservation, we must transform these equations to the Eulerian frame.
- Of course, we know that momentum cannot be preseved.
- Our purpose is to determine if energy is conserved.

The Comoving-Frame Diffusion Approximation

- The effective energy equation can be expressed as follows:

$$\partial_t E - \frac{2}{c^2} v_i \partial_t F_i + \partial_i F_i^e = c\sigma_a \left(aT^4 - E_0 \right) - \frac{\sigma_t}{c} v_i F_{0,i} \, , \qquad (26)$$

$$F_i^e = -\frac{c}{3\sigma_t} \partial_i E + \left[\frac{2}{3\sigma_t c} \partial_i \left(v_k F_k \right) \right] + \frac{4}{3} v_i E \, , \qquad (27)$$

  where F_i^e denotes the "effective" Eulerian-frame radiation energy flux.

- Equations (26) and (27) do not represent a conservative system because of the time-derivative of the flux in Eq.(26).

- This term is of the same type as terms that were set zero, but that does not mean that it is always negligible in highly non-equilibrium problems.

The Equilibrium-Diffusion Limit

- The diffusion-limit equations are derived from the radiation-hydrodynamic equations as follows.
  - Non-dimensionalize the equations.
  - Identify appropriate non-dimensional physical parameters.
  - Scale each physical parameter by an appropriate power of ϵ.
  - Return the scaled equations to dimensional form.
  - Expand each unknown as a power series in ϵ.
  - Expand all explicit functions of the unknowns in a power series in ϵ.
  - Substitute these series into the radiation-hydrodynamic equations.
  - Create a heirarchical system of equations for the unknown expansion coefficients by successively equating all terms multiplied by each power of ϵ.
  - Determine the equations solved by the leading-order coefficients.

The Equilibrium-Diffusion Limit

- The equilibrium-diffusion limit equations to leading order are as follows:

$$\partial_t \rho + \partial_i \left(\rho v_i\right) = 0 \,, \tag{28}$$

$$\partial_t \left(\rho v_i\right) + \partial_j \left(\rho v_i v_j\right) + \partial_i \left(p + \frac{1}{3} a T^4\right) = 0 \,, \tag{29}$$

$$\partial_t \left(\frac{1}{2}\rho v^2 + \rho e + a T^4\right) +$$

$$\partial_i \left[\left(\frac{1}{2}\rho v^2 + \rho e + p\right) v_i - \frac{c}{3\sigma_t} \partial_i a T^4 + \frac{4}{3} v_i a T^4\right] = 0 \,. \tag{30}$$

The Equilibrium-Diffusion Limit

- Continuing:

$$E = E_0 = aT^4 \,, \tag{31}$$

$$F_i = -\frac{c}{3\sigma_t}\partial_i aT^4 + \frac{4}{3}v_i aT^4 \,, \tag{32}$$

$$F_{i,0} = -\frac{c}{3\sigma_t}\partial_i aT^4 \,. \tag{33}$$

$$P_{ij} = P_{0,ij} = \frac{1}{3}\delta_{ij} aT^4 \,, \tag{34}$$

- Note that all radiation variables are explicit functions of the material temperature.

The Equilibrium-Diffusion Limit

- An equilibrium-diffusion limit expansion can be performed for any transport approximation.
- When such an expansion is performed for the Eulerian-frame and comoving-frame non-equilibrium grey diffusion equations, it is found that both of them preserve this limit *through* $O(\epsilon)$.
- This means that this limit is preserved with *full* accuracy by both approximations.
- This clearly indicates that there is nothing "inherently wrong" with the Eulerian-frame non-equilibrium grey diffusion approximation.
- The term that causes the lack of conservation in the comoving-frame diffusion equation disappears in the equilibrium-diffusion limit.

A Simplified Eulerian-Frame Approximation

- The comoving-frame grey diffusion equation is considerably simpler than the Eulerian-frame grey diffusion equation, but formally suffers from a lack of conservation.
- We now describe a simplification of the Eulerian-frame radiation momentum and energy sources that results in an Eulerian-frame diffusion equation with the same form as the comoving-frame equation.
- Let us begin by denoting the Eulerian-frame radiation momentum source by, $S_{m,i}$,

$$S_{m,i} = -\frac{\sigma_t}{c} F_{0,i} + \frac{v_i}{c} \sigma_a \left(aT^4 - E_0 \right) \ , \quad (35)$$

and the Eulerian-frame radiation energy source by, S_e,

$$S_e = \sigma_a \left(aT^4 - E_0 \right) - \frac{\sigma_t}{c} v_i F_{0,i} \ . \quad (36)$$

A Simplified Eulerian-Frame Approximation

- We simplfy these sources as follows:

$$S_{m,i} = -\frac{\sigma_t}{c} F_{0,i} \,, \tag{37}$$

$$S_e = \sigma_a \left(aT^4 - E \right) - \frac{\sigma_t}{c} v_i F_{0,i} \tag{38}$$

- This results in the following simplified Eulerian-frame diffusion equation:

$$\partial_t E - \partial_i \left(\frac{c}{3\sigma_t} \partial_i E \right) + \partial_i \left(v_i E \right) + \frac{1}{3} E \partial_i v_i = c\sigma_a \left(aT^4 - E \right) \,, \tag{39}$$

- Note that the simplified diffusion equation has exactly the same form as the comoving-frame diffusion equation but with E replacing E_0.

A Simplified Eulerian-Frame Approximation

- This simplified equation is not correct to O(v/c), but it nonetheless preserves equlibrium solutions, preserves the equilibrium-diffusion limit through first order, and is conservative.
- Using this equation is equivalent to using the comoving-frame diffusion equation, making a conservation statement with that equation (without transformation), and ignoring the difference between E_0 and E.
- This is the usual manner in which the comoving-frame equation is used.
- The derivation of the simplified equation adds justification to this standard but approximate approach in the sense that the consequences of the approximation are easier to understand.

Summary

- To obtain the grey diffusion approximation from the grey P_1 approximation, more terms must be neglected in the comoving frame than in the lab frame.
- The comoving-frame grey diffusion equation does not rigorously conserve the Eulerian-frame radiation energy to $O(v/c)$ when transformed to the Eulerian frame, but the error disappears in the asymptotic equilibrium diffusion limit.
- Because the comoving-frame P_1 equations conserve the Eulerian-frame radiation energy and the Eulerian-frame radiation momentum to $O(v/c)$ when transformed to the Eulerian frame, the lack of conservation in the diffusion approximation arises from terms that are dropped from the P_1 equations to obtain the diffusion approximation.

Summary

- Both the Eulerian-frame and comoving-frame grey diffusion approximations preserve the asymptotic equilibrium-diffusion limit through *first order*, and thus both yield full accuracy in this limit.
- The comoving-frame grey diffusion equation is considerably simpler than the Eulerian-frame diffusion equation.
- A simplification to the Eulerian-frame radiation energy and momentum source terms results in an Eulerian-frame grey diffusion equation that is equivalent to the comoving-frame equation with the Eulerian-frame radiation energy density replacing the comoving-frame radiation energy density.
- This simplified equation is not correct to O(v/c), but it nonetheless preserves equlibrium solutions, preserves the equilibrium-diffusion limit through first order, and is conservative.

Summary

- This equation adds justification to the usual approximate use of the comoving-frame diffusion equation in that the consequences are easier to understand.