Comoving-frame and Laboratory-frame Nonequilibrium Grey Radiation Diffusion Approximations in the Non-Relativistic Limit

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Motivation

• The purpose of this work is to contrast the non-equilibrium grey diffusion approximations in the Eulerian and comoving frames in the limit of non-relativistic material flow.

• We define the non-relativistic regime as $v/c < 0.01$.

• It has been suggested to us that any diffusion approximation in the Eulerian frame must be inherently flawed because “there is no diffusion in the Eulerian frame”.

• Our results indicate that this is not the case in the nonrelativistic limit.

• Indeed, the Eulerian-frame equation further justifies the approximate manner in which the comoving-frame diffusion approximation is usually used in the nonrelativistic limit.
Overview

- The fluid equations with grey radiation coupling to $O(v/c)$.
- The Eulerian-frame radiation energy and momentum equations, the $P_1$ equations, and the diffusion equation, to $O(v/c)$.
- The comoving-frame radiation energy and momentum equations, the $P_1$ equations, and the diffusion equation, to $O(v/c)$.
- The asymptotic equilibrium-diffusion limit.
- A simplified Eulerian-frame equation.
- Summary of results.
Fluid Equations with Radiation Coupling

- Conservation of fluid mass:
  \[ \partial_t \rho + \partial_i (\rho v_i) = 0, \]  
  \( (1) \)

- Conservation of fluid momentum:
  \[ \partial_t (\rho v_i) + \partial_j (\rho v_i v_j) + \partial_i p = \frac{\sigma_t}{c} F_{0,i} - \frac{v_i}{c} \sigma_a (aT^4 - E_0), \]  
  \( (2) \)

- Conservation of total fluid energy:
  \[ \partial_t \left( \frac{1}{2} \rho v^2 + \rho e \right) + \partial_i \left[ \left( \frac{1}{2} \rho v^2 + \rho e + p \right) v_i \right] = \]  
  \[ -c \sigma_a (aT^4 - E_0) + \frac{\sigma_t}{c} v_i F_{0,i}, \]  
  \( (3) \)

- where the radiation coupling is most naturally given in terms of comoving-frame quantities.
Fluid Equations with Radiation Coupling

• The Eulerian-to-comoving transformations to $O(v/c)$ are given by:

\[ E_0 = E - \frac{2}{c^2} v_i (F_i - v_i E - v_j P_{ij}) , \]  

\[ F_{0,i} = F_i - v_i E - v_j P_{ij} , \]  

\[ P_{0,i,j} = P_{i,j} - \frac{1}{c^2} (v_i F_j + F_i v_j) . \]  

• These relationships can be used to express the fluid-radiation coupling in terms of Eulerian-frame quantities.
The Eulerian-Frame Radiation Equations

• Conservation of radiation momentum:

\[
\frac{1}{c^2} \partial_t F_i + \partial_j P_{ij} = -\frac{\sigma_t}{c} F_{0,i} + \frac{v_i}{c} \sigma_a \left(aT^4 - E_0\right), \tag{7}
\]

• Conservation of radiation energy:

\[
\partial_t E + \partial_i F_i = c\sigma_a \left(aT^4 - E_0\right) - \frac{\sigma_t}{c} v_i F_{0,i}, \tag{8}
\]

• To obtain the P$_1$ equations, we need only set $P_{ij} = \frac{1}{3} \delta_{ij} E$ in the radiation momentum equation.
The Eulerian-Frame $P_1$ Equations

- Conservation of radiation momentum:

\[
\frac{1}{c^2} \partial_t F_i + \frac{1}{3} \partial_i E = -\frac{\sigma_t}{c} F_{0,i} + \frac{v_i}{c} \sigma_a \left( aT^4 - E_0 \right),
\]  

(9)

- Conservation of radiation energy:

\[
\partial_t E + \partial_i F_i = c\sigma_a \left( aT^4 - E_0 \right) - \frac{\sigma_t}{c} v_i F_{0,i},
\]  

(10)

where

\[
F_{0,i} = F_i - v_i \frac{4}{3} E.
\]  

(11)

- To obtain the diffusion approximation, we need only set the time-derivative of the flux to zero in the radiation momentum equation.
The Eulerian-Frame Diffusion Equation

- This causes momentum conservation to be lost, but energy is still conserved.
- Solving for the flux, we get

\[
F_i = -\frac{c}{3\sigma_t} \partial_i E + v_i \frac{4}{3} E + v_i \frac{\sigma_a}{\sigma_t} \left(aT^4 - E_0\right). \tag{12}
\]

- Substituting from Eq.(12) into Eq.(10), we obtain the Eulerian-frame grey diffusion equation:

\[
\partial_t E - \partial_i \frac{c}{3\sigma_t} \partial_i E + \partial_i \left\{ v_i \left[ \frac{4}{3} E + \frac{\sigma_a}{\sigma_t} \left(aT^4 - E_0\right) \right] \right\} =
\]
\[
c\sigma_a \left(aT^4 - E_0\right) - \frac{\sigma_t}{c} v_i F_{0,i}. \tag{13}
\]
The Comoving-Frame Radiation Equations

- The Radiation Momentum Equation:

\[
\frac{1}{c^2} \partial_t F_{0,i} + \frac{1}{c^2} v_j \partial_t P_{0,ij} + \partial_j P_{0,ij} + \frac{1}{c^2} F_{0,j} \partial_j v_i + \frac{1}{c^2} \partial_j (F_{0,i} v_j) = -\frac{\sigma_t}{c} F_{0,i} .
\]

(14)

- The Radiation Energy Equation:

\[
\partial_t E_0 + \frac{1}{c^2} v_i \partial_t F_{0,i} + \partial_i F_{0,i} + \partial_i (E_0 v_i) + \partial_i (P_{0,ij} v_j) - v_i \partial_j P_{0,ij} = c\sigma_a (aT^4 - E_0) .
\]

(15)

- To obtain the comoving-frame $P_1$ equations, we need simply assume that

\[
P_{0,ij} = \frac{1}{3} \delta_{ij} E_0 .
\]

(16)
The Comoving-Frame $P_1$ Equations

- The Radiation Momentum Equation:

$$\frac{1}{c^2} \partial_t F_{0,i} + \frac{1}{3c^2} v_i \partial_t E_0 + \frac{1}{3} \partial_i E_0 + \frac{1}{c^2} F_{0,j} \partial_j v_i + \frac{1}{c^2} \partial_j (F_{0,i} v_j) = - \frac{\sigma_t}{c} F_{0,i}.$$  \hspace{1cm} (17)

- The Radiation Energy Equation:

$$\partial_t E_0 + \frac{1}{c^2} v_i \partial_t F_{0,i} + \partial_i F_{0,i} + \partial_i (v_i E_0) + \frac{1}{3} E_0 \partial_i v_i = c\sigma_a \left(aT^4 - E_0\right).$$  \hspace{1cm} (18)

- Since the comoving frame is not inertial, we must transform these equations to the Eulerian frame to determine if they are conservative.
The Comoving-Frame $P_1$ Equations

- Making this transformation, we obtain the following momentum and energy equations:

\[
\frac{1}{c^2} \partial_t F_i + \partial_j P_{ij}^e = -\frac{\sigma_t}{c} F_{0,i} + \frac{v_i}{c} \sigma_a \left( a T^4 - E_0 \right), \tag{19}
\]

where

\[
P_{ij}^e = \frac{1}{3} E \delta_{ij} - \frac{1}{c^2} \left[ \frac{2}{3} (v_k F_k) \delta_{ij} - v_i F_j - F_i v_j \right]. \tag{20}
\]

\[
\partial_t E + \partial_i F_i = c \sigma_a \left( a T^4 - E_0 \right) - \frac{\sigma_t}{c} v_i F_{0,i}. \tag{21}
\]

- These equations are conservative, and the energy equation is actually identical to that of the Eulerian-frame $P_1$ equation.
The Comoving-Frame Diffusion Approximation

• To obtain the grey diffusion approximation, we must set all velocity dependent terms to zero in Eq.(17), and we must set the time derivative of the flux to zero in both Eqs. (17) and (18).

\[
\frac{1}{3} \partial_i E_0 = - \frac{\sigma_t}{c} F_{0,i},
\]

(22)

and

\[
\partial_t E_0 + \partial_i F_{0,i} + \partial_i (v_i E_0) + \frac{1}{3} E_0 \partial_i v_i = c\sigma_a (aT^4 - E_0).
\]

(23)

• Solving Eq.(22) for the flux, we get Fick’s law:

\[
F_{0,i} = - \frac{c}{3\sigma_t} \partial_i E_0.
\]

(24)
The Comoving-Frame Diffusion Approximation

• Substituting from Eq.(24) into Eq.(23), we get the following diffusion equation for the radiation energy density:

\[
\partial_t E_0 - \partial_i \left( \frac{c}{3\sigma_t} \partial_i E_0 \right) + \partial_i (v_i E_0) + \frac{1}{3} E_0 \partial_i v_i = c\sigma_a \left( aT^4 - E_0 \right). \tag{25}
\]

• To investigate conservation, we must transform these equations to the Eulerian frame.

• Of course, we know that momentum cannot be preserved.

• Our purpose is to determine if energy is conserved.
The Comoving-Frame Diffusion Approximation

- The effective energy equation can be expressed as follows:

\[
\frac{\partial}{\partial t} E - \frac{2}{c^2} v_i \frac{\partial}{\partial t} F_i + \frac{\partial}{\partial i} F_{ie} = c\sigma a (aT^4 - E_0) - \frac{\sigma}{c} v_i F_{0,i},
\]

(26)

\[
F_{ie} = -\frac{c}{3\sigma_t} \frac{\partial}{\partial i} E + \left[ \frac{2}{3\sigma_t} \frac{\partial}{\partial i} (v_k F_k) \right] + \frac{4}{3} v_i E,
\]

(27)

where \(F_{ie}\) denotes the “effective” Eulerian-frame radiation energy flux.

- Equations (26) and (27) do not represent a conservative system because of the time-derivative of the flux in Eq.(26).

- This term is of the same type as terms that were set zero, but that does not mean that it is always negligible in highly non-equilibrium problems.
The Equilibrium-Diffusion Limit

- The diffusion-limit equations are derived from the radiation-hydrodynamic equations as follows.
  - Non-dimensionalize the equations.
  - Identify appropriate non-dimensional physical parameters.
  - Scale each physical parameter by an appropriate power of $\epsilon$.
  - Return the scaled equations to dimensional form.
  - Expand each unknown as a power series in $\epsilon$.
  - Expand all explicit functions of the unknowns in a power series in $\epsilon$.
  - Substitute these series into the radiation-hydrodynamic equations.
  - Create a hierarchical system of equations for the unknown expansion coefficients by successively equating all terms multiplied by each power of $\epsilon$.
  - Determine the equations solved by the leading-order coefficients.
The Equilibrium-Diffusion Limit

- The equilibrium-diffusion limit equations to leading order are as follows:

\[ \partial_t \rho + \partial_i (\rho v_i) = 0 , \]  
\[ \partial_t (\rho v_i) + \partial_j (\rho v_i v_j) + \partial_i \left( p + \frac{1}{3} a T^4 \right) = 0 , \]  
\[ \partial_t \left( \frac{1}{2} \rho v^2 + \rho e + a T^4 \right) + \partial_i \left[ \left( \frac{1}{2} \rho v^2 + \rho e + p \right) v_i - \frac{c}{3 \sigma_t} \partial_i a T^4 + \frac{4}{3} v_i a T^4 \right] = 0 . \]
The Equilibrium-Diffusion Limit

• Continuing:

\[ E = E_0 = aT^4, \quad (31) \]

\[ F_i = -\frac{c}{3\sigma_t} \partial_i aT^4 + \frac{4}{3} v_i aT^4, \quad (32) \]

\[ F_{i,0} = -\frac{c}{3\sigma_t} \partial_i aT^4. \quad (33) \]

\[ P_{ij} = P_{0,ij} = \frac{1}{3} \delta_{ij} aT^4, \quad (34) \]

• Note that all radiation variables are explicit functions of the material temperature.
The Equilibrium-Diffusion Limit

• An equilibrium-diffusion limit expansion can be performed for any transport approximation.

• When such an expansion is performed for the Eulerian-frame and comoving-frame non-equilibrium grey diffusion equations, it is found that both of them preserve this limit through $O(\epsilon)$.

• This means that this limit is preserved with full accuracy by both approximations.

• This clearly indicates that there is nothing “inherently wrong” with the Eulerian-frame non-equilibrium grey diffusion approximation.

• The term that causes the lack of conservation in the comoving-frame diffusion equation disappears in the equilibrium-diffusion limit.
A Simplified Eulerian-Frame Approximation

- The comoving-frame grey diffusion equation is considerably simpler than the Eulerian-frame grey diffusion equation, but formally suffers from a lack of conservation.

- We now describe a simplification of the Eulerian-frame radiation momentum and energy sources that results in an Eulerian-frame diffusion equation with the same form as the comoving-frame equation.

- Let us begin by denoting the Eulerian-frame radiation momentum source by, $S_{m,i}$,

$$ S_{m,i} = \frac{\sigma_t}{c} F_{0,i} + \frac{v_i}{c} \sigma_a \left( aT^4 - E_0 \right) , \quad (35) $$

and the Eulerian-frame radiation energy source by, $S_e$,

$$ S_e = \sigma_a \left( aT^4 - E_0 \right) - \frac{\sigma_t}{c} v_i F_{0,i} . \quad (36) $$
A Simplified Eulerian-Frame Approximation

- We simplify these sources as follows:

\[ S_{m,i} = -\frac{\sigma_t}{c} F_{0,i} \]  \hspace{1cm} (37)

\[ S_e = \sigma_a (aT^4 - E) - \frac{\sigma_t}{c} v_i F_{0,i} \] \hspace{1cm} (38)

- This results in the following simplified Eulerian-frame diffusion equation:

\[ \partial_t E - \partial_i \left( \frac{c}{3\sigma_t} \partial_i E \right) + \partial_i (v_i E) + \frac{1}{3} E \partial_i v_i = c\sigma_a \left( aT^4 - E \right) \] \hspace{1cm} (39)

- Note that the simplified diffusion equation has exactly the same form as the comoving-frame diffusion equation but with \( E \) replacing \( E_0 \).
• This simplified equation is not correct to $O(v/c)$, but it nonetheless preserves equilibrium solutions, preserves the equilibrium-diffusion limit through first order, and is conservative.

• Using this equation is equivalent to using the comoving-frame diffusion equation, making a conservation statement with that equation (without transformation), and ignoring the difference between $E_0$ and $E$.

• This is the usual manner in which the comoving-frame equation is used.

• The derivation of the simplified equation adds justification to this standard but approximate approach in the sense that the consequences of the approximation are easier to understand.
Summary

• To obtain the grey diffusion approximation from the grey $P_1$ approximation, more terms must be neglected in the comoving frame than in the lab frame.

• The comoving-frame grey diffusion equation does not rigorously conserve the Eulerian-frame radiation energy to $O\left(\frac{v}{c}\right)$ when transformed to the Eulerian frame, but the error disappears in the asymptotic equilibrium diffusion limit.

• Because the comoving-frame $P_1$ equations conserve the Eulerian-frame radiation energy and the Eulerian-frame radiation momentum to $O\left(\frac{v}{c}\right)$ when transformed to the Eulerian frame, the lack of conservation in the diffusion approximation arises from terms that are dropped from the $P_1$ equations to obtain the diffusion approximation.
Summary

- Both the Eulerian-frame and comoving-frame grey diffusion approximations preserve the asymptotic equilibrium-diffusion limit through first order, and thus both yield full accuracy in this limit.
- The comoving-frame grey diffusion equation is considerably simpler than the Eulerian-frame diffusion equation.
- A simplification to the Eulerian-frame radiation energy and momentum source terms results in an Eulerian-frame grey diffusion equation that is equivalent to the comoving-frame equation with the Eulerian-frame radiation energy density replacing the comoving-frame radiation energy density.
- This simplified equation is not correct to $O\left(\frac{v}{c}\right)$, but it nonetheless preserves equilibrium solutions, preserves the equilibrium-diffusion limit through first order, and is conservative.
Summary

• This equation adds justification to the usual approximate use of the comoving-frame diffusion equation in that the consequences are easier to understand.