Radiative Shock Solutions

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Aside from “hydro + heat conduction,” no analytic solutions exist for rad-hydro.

- Generate *semi-analytic* solutions; solutions from numerically integrating nonlinear ODEs.
- Seek traveling wave solutions ⇒ *radiative shocks*.

Use solutions to

- verify code correctness,
- test material motion corrections,
- test AMR (rad-shocks are multiscale problems),
- evaluate time integration splitting effects.

Gives physics insight.
Outline

1. Review structure of radiative shocks

2. Equations of radiation hydrodynamics
   - Euler coupled with grey nonequilibrium diffusion
   - Equilibrium diffusion limit
   - Reduced equations (ODEs; jump relations)

3. Sample Solutions

4. Approximations

5. Code Comparison
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Regimes in astrophysics governed by

- Inviscid hydrodynamics
- Thermal radiation (X-ray) transport
- High-energy density:
  - Material temperatures $O(1\text{KeV})$
  - Radiation pressure affects hydrodynamics
Past Work on Radiative Shocks

- **Overview of Theory:**


- **Theory and Solutions:**
Inviscid Hydrodynamic Shocks

Shock Jump

\( T_1 \)

\( T_0 \)

\( T \)

Temperature

\( x \)

Radiative Shock Solutions
Subcritical Radiative Shocks \((T_p < T_1)\)
Supercritical Radiative Shocks ($T_p = T_1$)
Zel’дович Spike

Hydro Shock (Zel’dovich Spike)

Relaxation Region

\( T_p = T_1 \)

Lowrie Radiative Shock Solutions

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Euler Coupled with Grey Nonequilibrium Diffusion

Nondimensional form, 1-D planar (slab) geometry

Most of this talk will be about solutions of

\[ \partial_t \rho + \partial_x (\rho v) = 0, \]
\[ \partial_t (\rho v) + \partial_x \left( \rho v^2 + p + \frac{1}{3} P_0 \theta^4 \right) = 0, \]
\[ \partial_t (\rho E) + \partial_x \left[ v (\rho E + p) \right] = P_0 \sigma_a (\theta^4 - T^4) - \frac{1}{3} P_0 v \partial_x \theta^4, \]
\[ \partial_t (\rho E + P_0 \theta^4) + \partial_x \left[ v \left( \rho E + p + \frac{4}{3} P_0 \theta^4 \right) \right] = P_0 \partial_x (\kappa \partial_x \theta^4), \]

where \( E = e + \frac{1}{2} v^2 \) and for a \( \gamma \)-law EOS

\[ p = \frac{\rho T}{\gamma}, \quad e = \frac{T}{\gamma(\gamma - 1)}, \quad P_0 = \frac{\tilde{\alpha} R \tilde{T}_0^4}{\gamma \tilde{\rho}_0} \approx \text{rad. pressure}, \quad \text{mat. pressure}. \]

Equilibrium Diffusion Limit

We’ll also discuss the “1-T limit.” Optically thick limit, $\theta \rightarrow T$, and our system reduces to

$$\partial_t \rho + \partial_x (\rho v) = 0,$$

$$\partial_t (\rho v) + \partial_x \left( \rho v^2 + p + \frac{1}{3} P_0 T^4 \right) = 0,$$

$$\partial_t (\rho E + P_0 T^4) + \partial_x \left[ v \left( \rho E + p + \frac{4}{3} P_0 T^4 \right) \right] = P_0 \partial_x (\kappa \partial_x T^4).$$

Shock solutions found in Lowrie & Rauenzahn (2007).
Problem statement

- Assume that far from shock, $\theta = T$.
- Galilean invariant, so use steady frame.

**Given:** The value $\gamma$ and
- Pre-shock state ($x \to -\infty$): $\rho_0$, $T_0$, $\theta_0 = T_0$.
- Non-dimensional constants: $P_0$ and $M_0$ ($\equiv \tilde{v}_0/\tilde{a}_0$, Mach number).
- Functions $\sigma_a(\rho, T)$ and $\kappa(\rho, T)$ ($\tilde{\kappa} = \tilde{c}/3\tilde{\sigma}_t$).

**Calculate:** $\rho(x)$, $v(x)$, $T(x)$, and $\theta(x)$.

**Optional:** Transform back to a frame where shock is moving.
Integrate $-\infty < x < \infty$ to give:

\[
\begin{pmatrix}
\rho v \\
\rho v^2 + p^* \\
(\rho E^* + p^*)v
\end{pmatrix}_0 = \begin{pmatrix}
\rho v \\
\rho v^2 + p^* \\
(\rho E^* + p^*)v
\end{pmatrix}_1,
\]

where

\[
p^* = p + \frac{1}{3} \mathcal{P}_0 T^4, \quad e^* = e + \frac{1}{\rho} \mathcal{P}_0 T^4, \quad E^* = e^* + \frac{1}{2} v^2.
\]

Get ninth-order polynomial in $T_1$; see Bouquet et al (2000) and Lowrie & Rauenzahn (2007).
Hydro Shock Relations

At a discontinuity separating state-$p$ and state-$s$:

$$
\begin{pmatrix}
\rho v \\
\rho v^2 + p \\
(\rho E + p)v \\
-\kappa \partial_x \theta^4 + \frac{4}{3} v \theta^4
\end{pmatrix}_p = \begin{pmatrix}
\rho v \\
\rho v^2 + p \\
(\rho E + p)v \\
-\kappa \partial_x \theta^4 + \frac{4}{3} v \theta^4
\end{pmatrix}_s
$$

- First 3 equations: Standard hydro jump conditions.
- Last equation: Continuity of Eulerian frame flux. $\theta$ continuous.
The 4 PDEs reduce to 2 ODEs:

\[
T' = \mathcal{P}_0 \frac{\nu 4 \theta^3 \theta' + 3 \sigma_a (\gamma \mathcal{M}^2 - 1)(\theta^4 - T^4)}{3 C_p \mathcal{M}_0 (\mathcal{M}^2 - 1)},
\]

\[
\theta' = \nu \frac{6 C_p \rho (T - 1) + 3 \rho (\nu^2 - \mathcal{M}_0^2) + 8 \mathcal{P}_0 (\theta^4 - \rho)}{24 \mathcal{P}_0 \kappa \theta^3}
\]

where \((\cdot)' = d(\cdot)/dx\), \(C_p = 1/(\gamma - 1)\), \(\mathcal{M} = \nu/\sqrt{T}\), \(\nu = \mathcal{M}_0/\rho\), and

\[
\rho(T, \theta) = \frac{K_m - \gamma \mathcal{P}_0 \theta^4 \pm \sqrt{(K_m - \gamma \mathcal{P}_0 \theta^4)^2 - 36 \gamma \mathcal{M}_0^2 T \nu^2}}{6 T}
\]

and \(K_m\) is an integration constant. **Issues:**

- Requires choice of root (or branch). Branch point at \(\mathcal{M} = 1/\sqrt{\gamma}\).
- Singular whenever \(\mathcal{M} = 1\).
Using $\mathcal{M}$ as the Independent Variable

Alternative to integrating $T'$, $\theta'$:

$$\frac{dx}{d\mathcal{M}} = 3\mathcal{M}_0(\mathcal{M}^2 - 1)\rho\beta$$

$$\frac{dT}{d\mathcal{M}} = (\gamma - 1)\beta \left[ 4\mathcal{M}_0\theta^3\theta' + (\gamma\mathcal{M}^2 - 1)r \right],$$

where $\beta(T, \mathcal{M})$ and $r(T, \mathcal{M})$ are known functions, and

$$\rho(T, \mathcal{M}) = \frac{\mathcal{M}_0}{\mathcal{M}\sqrt{T}}$$

$$\theta(T, \mathcal{M}) = \frac{1}{\gamma P_0} \left[ K_m - 3\gamma \frac{\mathcal{M}_0^2}{\rho(T, \mathcal{M})} - 3 T \rho(T, \mathcal{M}) \right]$$

- Branch points no longer an issue.
- Nonsingular at $\mathcal{M} = 1$. 
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Solution Regimes

Solutions are characterized by the following:

- Is there an embedded hydro shock?
  - If no, all variables continuous.
- Is there a branch point ($\mathcal{M} = 1/\sqrt{\gamma}$)?
Isothermal Shock ($1-T$) ⇔ Branch Point ($2-T$) Regions

$M_0 = 1.2$ solution

$P_0 = 10^{-4}$, $\sigma_a = 10^6$, $\kappa = 1$, $\gamma = 5/3$; subcritical; shock, but no branch point.
$\mathcal{M}_0 = 2$ solution

Subcritical; branch point coincident with shock.
\( \mathcal{M}_0 = 3 \) solution

Subcritical; branch point downstream of shock.
$M_0 = 3$ solution

Spike region.
\( M_0 = 5 \) solution

Supercritical; branch point downstream of shock.
$M_0 = 5$ solution (continued)
Zel’dovich spike region

Temperature

Density
$M_0 = 27$ solution

Supercritical; branch point downstream of shock. NOTE OVER COMPRESSION.
\( \mathcal{M}_0 = 27 \) solution (continued)

Zel’dovich spike region
$M_0 = 30$ solution

No shock, but still a branch point!
$M_0 = 30$ solution (continued)

Spike region

![Temperature and Density Graphs]

- Temperature graph shows the parameters $T$ and $T(1-T)$.
- Density graph shows the parameters $\rho$ and $\rho(1-T)$.
$\mathcal{M}_0 = 50$ solution

No shock or branch point
\( M_0 = 5 \), variable cross-sections (Bremsstrahlung)

\( P_0 = 10^{-4}, \gamma = 5/3, \kappa = 0.00175 T^{7/2}/\rho, \sigma_a = 10^6/\kappa \)
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Branch Point at \( T_{\text{max}} \)

- Assuming \( \theta \approx \) constant in spike region, can prove that \( T_{\text{max}} \) occurs at the branch point (\( \mathcal{M} = 1/\sqrt{\gamma} \)).

- Previous work assumes \( T \) decreases monotonically from hydro shock. Ignores “\( p \nabla \cdot \vec{v} \)” contribution.
Estimates for $T_{\text{max}}$

- Mihalas & Mihalas (1984) estimate:
  \[ T_{\text{max}} = (3 - \gamma) T_1 \]

- Our new estimate:
  \[ T_{\text{max}} = \begin{cases} \frac{1}{36\gamma M_0^2} \left[ 3(\gamma M_0^2 + 1) + \gamma P_0(1 - T_1^4) \right]^2 & \text{if } M_1 < 1/\sqrt{\gamma}, \\ T_1 & \text{otherwise.} \end{cases} \]

- Neither estimate requires detailed knowledge of profile; independent of cross-sections.
Comparison of $T_{\text{max}}$ Estimates

$P_0 = 10^{-4}$, $\gamma = 5/3$; exact also used $\sigma_a = 10^6$, $\kappa = 1$
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RAGE code (Gittings et al 2006):
- Cell-centered, high-resolution finite volume.
- Operator split: Hydro explicit, radiation implicit
- Uses adaptive mesh refinement (AMR)

Run $M = 5$ test case:
- Initialize with exact shock profile
- How well can RAGE propagate the shock and maintain profile?
- Use AMR with

$$\frac{\Delta x_{\text{max}}}{\Delta x_{\text{min}}} = 2^{L-1}, \quad L \in \{8, 10, 12, 13\}$$

For $L = 13$, used 967 cells, equivalent equally-spaced mesh is 204,800 cells.
$M_0 = 5$ RAGE Results

$\Delta x_{\text{max}} = 0.01$, shock propagated $\Delta x = 0.3$
$M_0 = 5$ RAGE Results (continued)

Zel’dovich spike region

\[ T \]

- Exact
- 8 levels
- 10 levels
- 12 levels
- 13 levels
Summary

- Semi-analytic solutions for equilibrium and nonequilibrium, radiative shocks.
- Temperature may increase after hydro shock.
  - Classic picture: \( T_{\text{max}} \) occurs at hydro shock.
  - New picture: \( T_{\text{max}} \) occurs at branch point.
- Derived very accurate estimate for \( T_{\text{max}} \); independent of radiation model and cross sections.
- Solutions may be used for code verification.

Future work
- Separate ion/electron temperatures
- Spherical symmetry
- More advanced radiation models
Overview of Nonequilibrium Solution Procedure

1. Compute post-shock state ($x \to \infty$): Find root of a ninth-order polynomial in $T_1$.
2. Find pre-shock solution: Integrate ODEs from $\mathcal{M} = \mathcal{M}_0$ to $\mathcal{M} = 1$.
3. Find post-shock solution: Integrate ODEs from $\mathcal{M} = \mathcal{M}_1$ to $\mathcal{M} = 1$.
4. Connect the pre- and post-shock branches with a hydro shock.
Error in new $T_{\text{max}}$ estimate

$\mathcal{P}_0 = 10^{-4}$, $\sigma_a = 10^6$, $\kappa = 1$, $\gamma = 5/3$