Radiative Shock Solutions

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Computational Physics (CCS-2)

Numerical Methods for Multi-Material Fluid Flows
10-14 September 2007
Prague, CZ

LA-UR-07-5986

Motivation

- Aside from "hydro + heat conduction," no analytic solutions exist for rad-hydro.
  - Generate *semi-analytic* solutions; solutions from numerically integrating nonlinear ODEs.
  - Seek traveling wave solutions $\Rightarrow$ *radiative shocks*.
- Use solutions to
  - verify code correctness,
  - test material motion corrections,
  - test AMR (rad-shocks are multiscale problems),
  - evaluate time integration splitting effects.
- Gives physics insight.

Outline

1. Review structure of radiative shocks
2. Equations of radiation hydrodynamics
   - Euler coupled with grey nonequilibrium diffusion
   - Equilibrium diffusion limit
   - Reduced equations (ODEs; jump relations)
3. Sample Solutions
4. Approximations
5. Code Comparison

Outline

Physics Regime

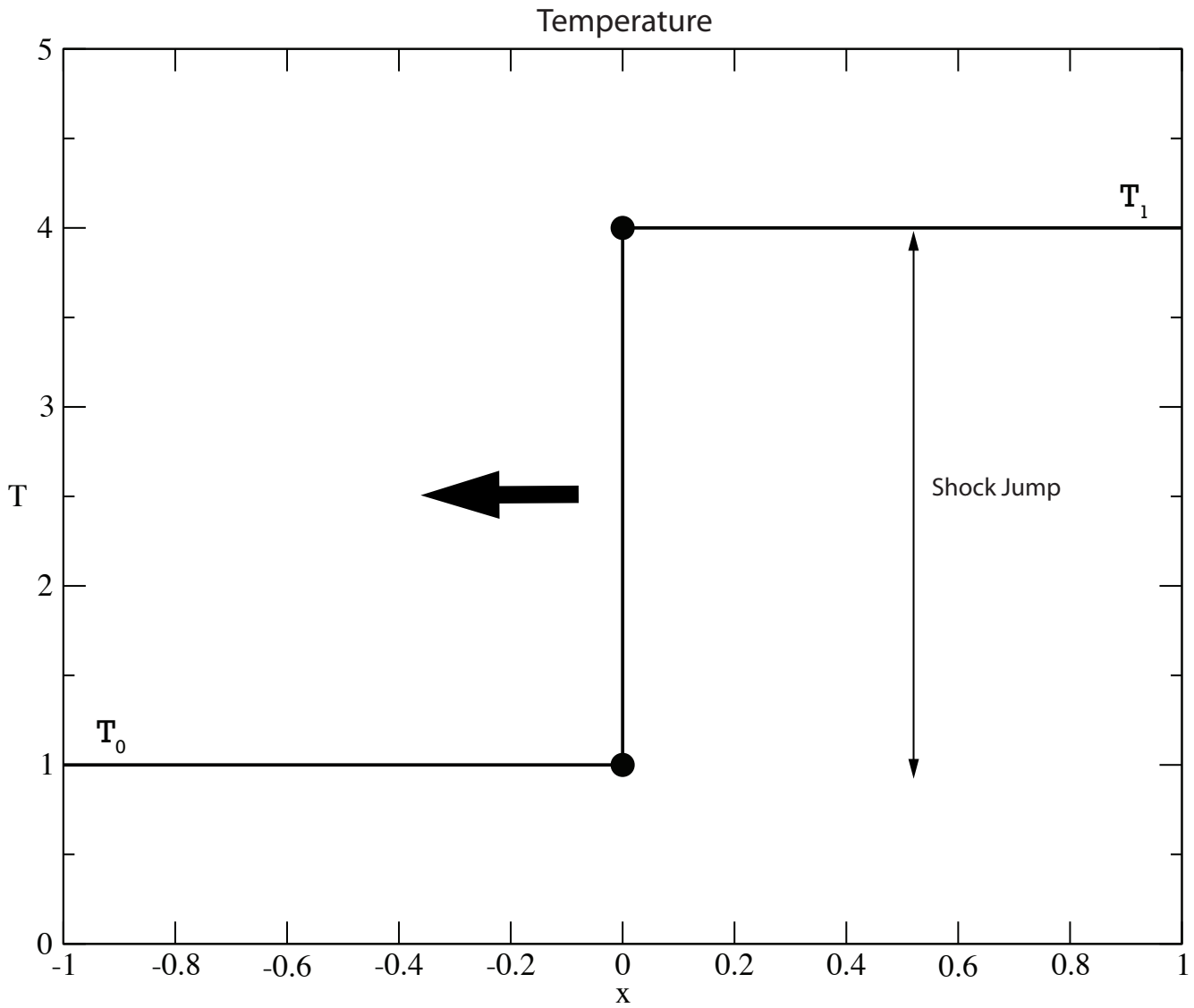
Regimes in astrophysics governed by

- Inviscid hydrodynamics
- Thermal radiation (X-ray) transport
- High-energy density:
  - Material temperatures $O(1 KeV)$
  - Radiation *pressure* affects hydrodynamics

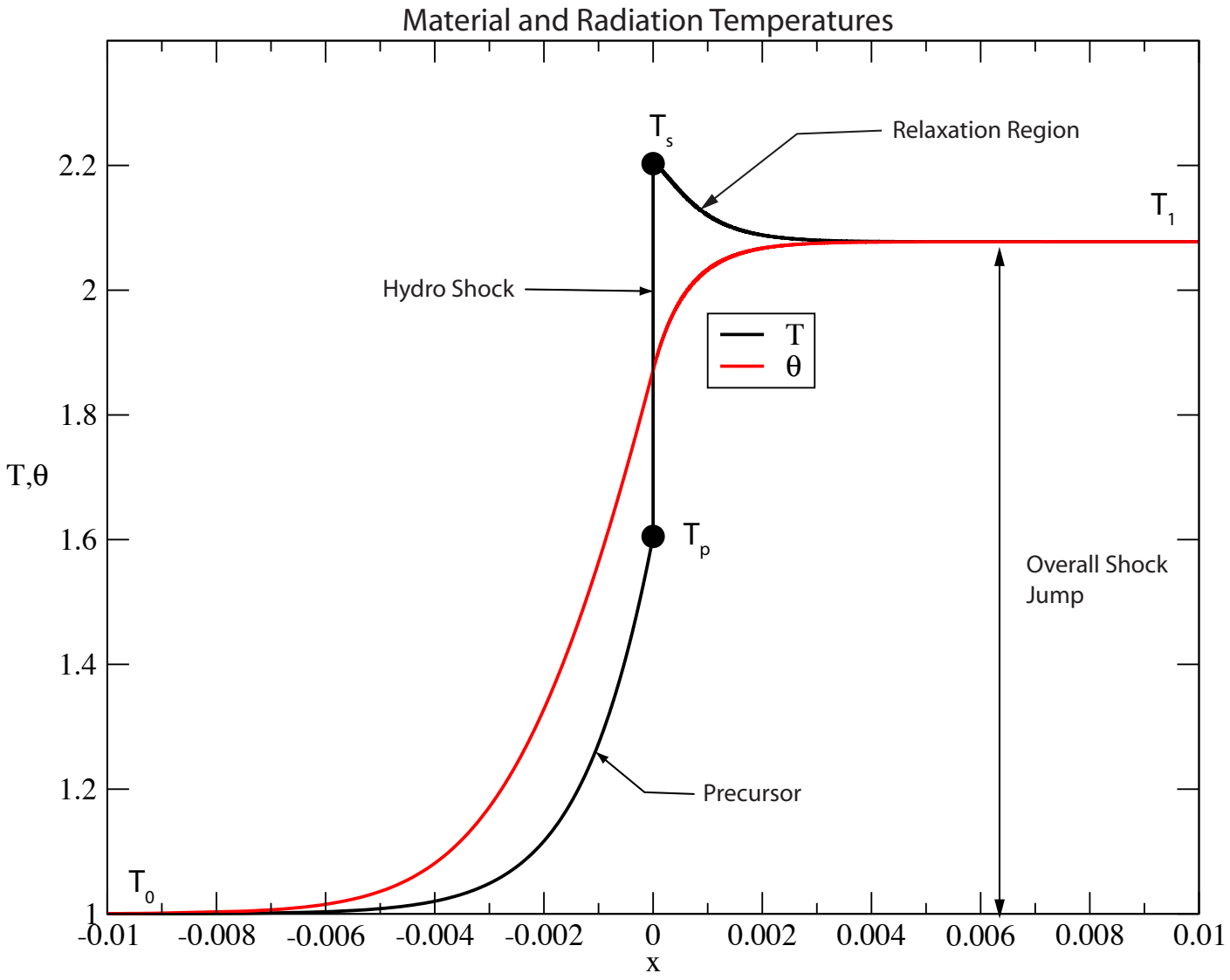
Past Work on Radiative Shocks

- **Overview of Theory:**
  - Y. B. ZEL'DOVICH and Y. P. RAIZER, *Physics of Shock Waves and High-Temperature Hydrodynamic Phenomena*, 1966 (Dover 2002).
  - D. MIHALAS and B. W. MIHALAS, *Foundations of Radiation Hydrodynamics*, 1984 (Dover 1999).
- **2-T and 3-T VEF/AMR calculations:** M. W. SINCELL, M. GEHMEYR, and D. MIHALAS, two articles in *Shock Waves*, 1999.
- **Theory and Solutions:**
  - **1-T:** S. BOUQUET, R. TEYSSIER, and J. P. CHIEZE, *Astrophysical Journal Supplement Series*, 2000.
  - **1-T:** R. B. LOWRIE and R. M. RAUENZAHN, *Shock Waves*, 2007.
  - **2-T:** R. B. LOWRIE and J. D. EDWARDS, *Proceedings of ANS Computations + Supercomputing in Nuclear Applications 2007 Joint Topical Meeting.* Follow-up in preparation.

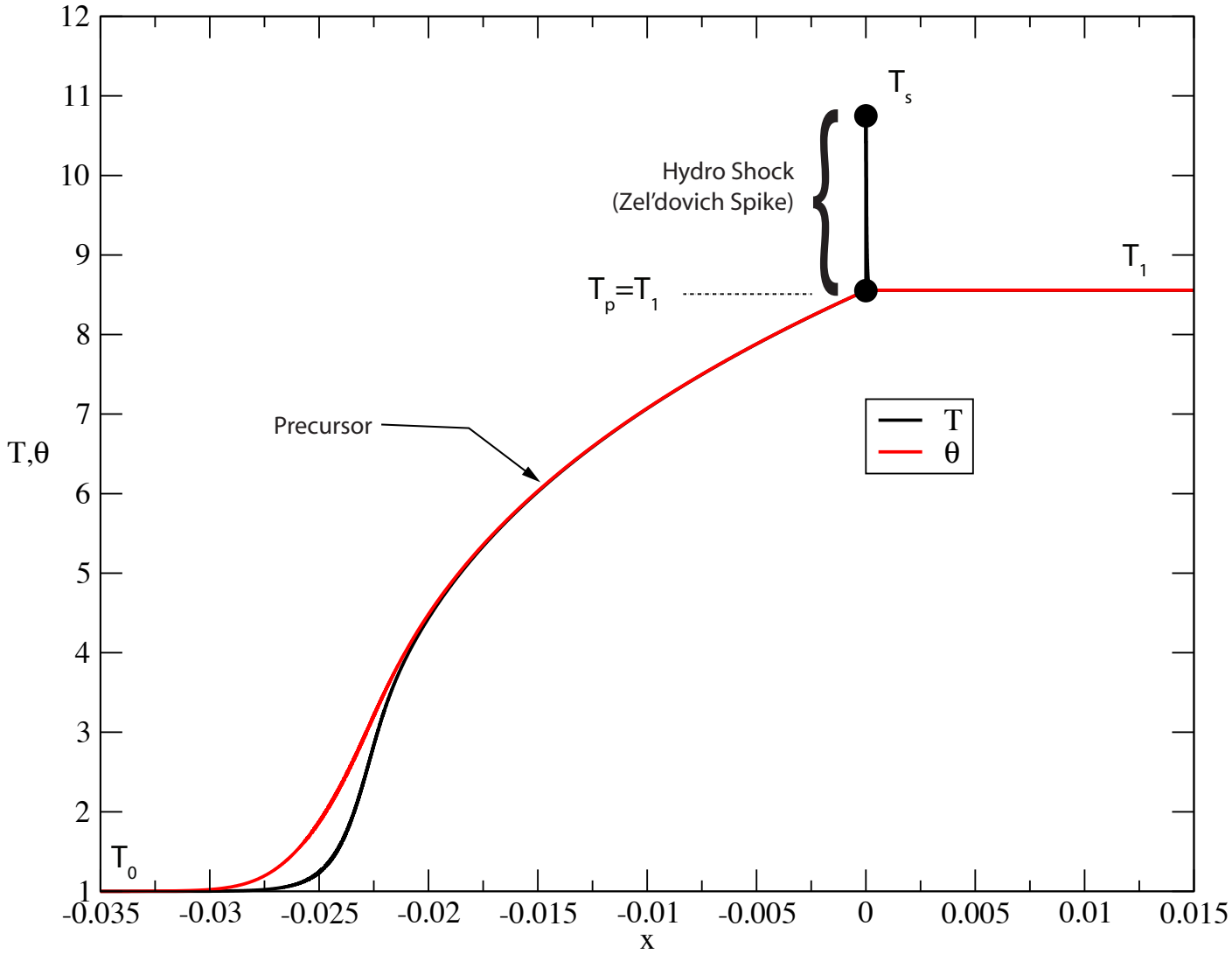
Inviscid Hydrodynamic Shocks



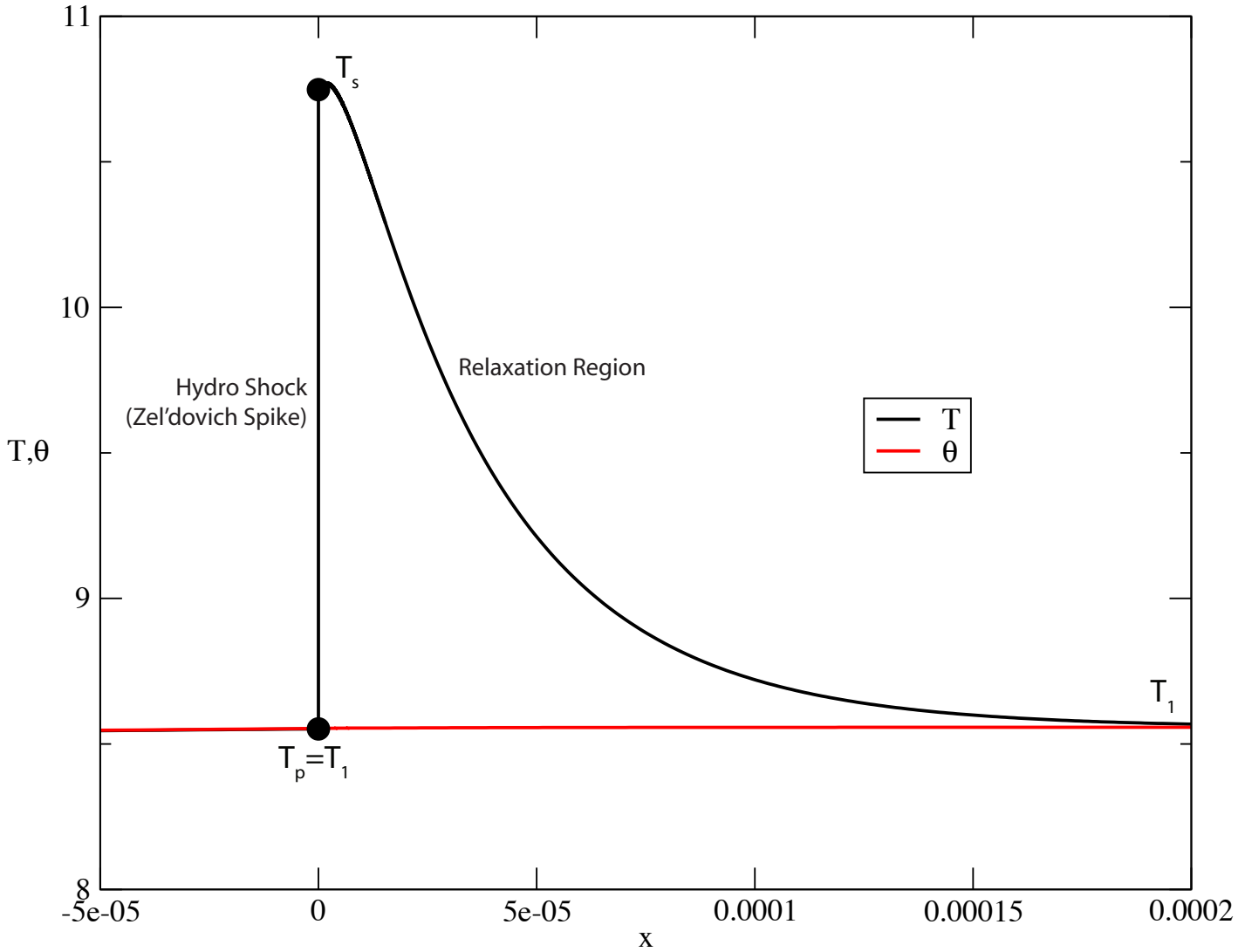
Subcritical Radiative Shocks ($T_p < T_1$)



Supercritical Radiative Shocks ($T_p = T_1$)



Zel'dovich Spike



Outline

Euler Coupled with Grey Nonequilibrium Diffusion

Nondimensional form, 1-D planar (slab) geometry

Most of this talk will be about solutions of

$$\partial_t \rho + \partial_x(\rho v) = 0,$$

$$\partial_t(\rho v) + \partial_x\left(\rho v^2 + p + \frac{1}{3}\mathcal{P}_0\theta^4\right) = 0,$$

$$\partial_t(\rho E) + \partial_x\left[v(\rho E + p)\right] = \mathcal{P}_0\sigma_a(\theta^4 - T^4) - \frac{1}{3}\mathcal{P}_0 v\partial_x\theta^4,$$

$$\partial_t(\rho E + \mathcal{P}_0\theta^4) + \partial_x\left[v\left(\rho E + p + \frac{4}{3}\mathcal{P}_0\theta^4\right)\right] = \mathcal{P}_0\partial_x(\kappa\partial_x\theta^4),$$

where $E = e + \frac{1}{2}v^2$ and for a γ-law EOS

$$p = \frac{\rho T}{\gamma}, \quad e = \frac{T}{\gamma(\gamma-1)}, \quad \mathcal{P}_0 = \frac{\tilde{\alpha}_R \tilde{T}_0^4}{\gamma\tilde{p}_0} \approx \frac{\text{rad. pressure}}{\text{mat. pressure}}.$$

Shock solutions found in Lowrie & Edwards (2007).

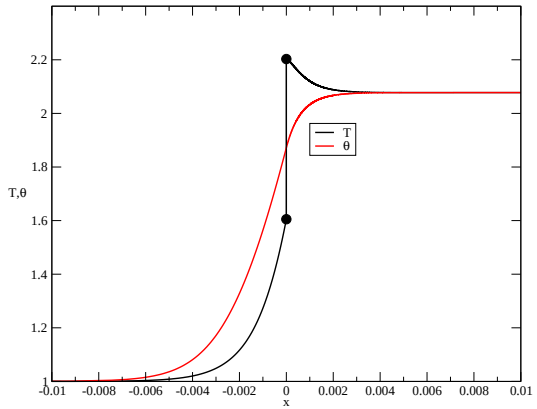
Equilibrium Diffusion Limit

We'll also discuss the "1-T limit." Optically thick limit, $\theta \to T$, and our system reduces to

$$\partial_t \rho + \partial_x(\rho v) = 0,$$
$$\partial_t(\rho v) + \partial_x\left(\rho v^2 + p + \frac{1}{3}\mathcal{P}_0 T^4\right) = 0,$$
$$\partial_t(\rho E + \mathcal{P}_0 T^4) + \partial_x\left[v\left(\rho E + p + \frac{4}{3}\mathcal{P}_0 T^4\right)\right] = \mathcal{P}_0 \partial_x(\kappa \partial_x T^4).$$

Shock solutions found in Lowrie & Rauenzahn (2007).

Problem statement



- Assume that far from shock, $\theta = T$.
- Galilean invariant, so use steady frame.

- *Given:* The value γ and
  - Pre-shock state ($x \to -\infty$): ρ_0, T_0, $\theta_0 = T_0$.
  - Non-dimensional constants: $\mathcal{P}_0$ and $\mathcal{M}_0$ ($\equiv \tilde{v}_0/\tilde{a}_0$, Mach number).
  - Functions $\sigma_a(\rho, T)$ and $\kappa(\rho, T)$ ($\tilde{\kappa} = \tilde{c}/3\tilde{\sigma}_t$).
- *Calculate:* $\rho(x)$, $v(x)$, $T(x)$, and $\theta(x)$.
- *Optional:* Transform back to a frame where shock is moving.

Overall Jump Relation

Integrate $-\infty < x < \infty$ to give:

$$\begin{pmatrix} \rho v \\ \rho v^2 + p^* \\ (\rho E^* + p^*) v \end{pmatrix}_0 = \begin{pmatrix} \rho v \\ \rho v^2 + p^* \\ (\rho E^* + p^*) v \end{pmatrix}_1 ,$$

where

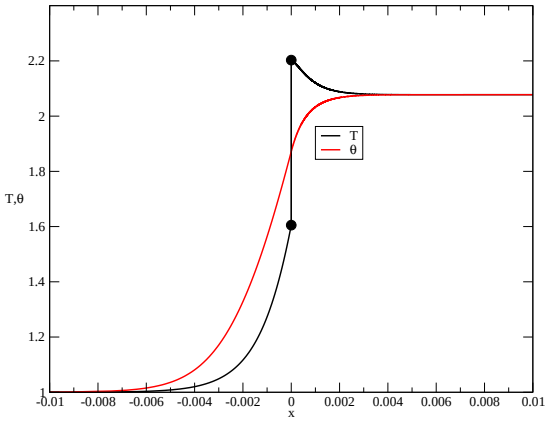
$$p^* = p + \frac{1}{3}\mathcal{P}_0 T^4, \quad e^* = e + \frac{1}{\rho}\mathcal{P}_0 T^4, \quad E^* = e^* + \frac{1}{2}v^2.$$

Get ninth-order polynomial in T_1; see Bouquet *et al* (2000) and Lowrie & Rauenzahn (2007).

Hydro Shock Relations

At a discontinuity separating state-$_p$ and state-$_s$:

$$\begin{pmatrix} \rho v \\ \rho v^2 + p \\ (\rho E + p) v \\ -\kappa \partial_x \theta^4 + \frac{4}{3} v \theta^4 \end{pmatrix}_p = \begin{pmatrix} \rho v \\ \rho v^2 + p \\ (\rho E + p) v \\ -\kappa \partial_x \theta^4 + \frac{4}{3} v \theta^4 \end{pmatrix}_s$$



- First 3 equations: Standard hydro jump conditions.
- Last equation: Continuity of *Eulerian frame* flux. θ continuous.

Governing Ordinary Differential Equations

The 4 PDEs reduce to 2 ODEs:

$$T' = \mathcal{P}_0 \frac{v 4\theta^3\theta' + 3\sigma_a(\gamma\mathcal{M}^2 - 1)(\theta^4 - T^4)}{3C_p\mathcal{M}_0(\mathcal{M}^2 - 1)},$$

$$\theta' = v\frac{6C_p\rho(T-1) + 3\rho(v^2 - \mathcal{M}_0^2) + 8\mathcal{P}_0(\theta^4 - \rho)}{24\mathcal{P}_0\kappa\theta^3}$$

where $(\cdot)' = d(\cdot)/dx$, $C_p = 1/(\gamma - 1)$, $\mathcal{M} = v/\sqrt{T}$, $v = \mathcal{M}_0/\rho$, and

$$\rho(T,\theta) = \frac{K_m - \gamma\mathcal{P}_0\theta^4 \pm \sqrt{(K_m - \gamma\mathcal{P}_0\theta^4)^2 - 36\gamma\mathcal{M}_0^2 T}}{6T}$$

and K_m is an integration constant. Issues:

- Requires choice of root (or branch). Branch point at $\mathcal{M} = 1/\sqrt{\gamma}$.
- Singular whenever $\mathcal{M} = 1$.

Using $\mathcal{M}$ as the Independent Variable

Alternative to integrating T', θ':

$$\frac{dx}{d\mathcal{M}} = \frac{3\mathcal{M}_0(\mathcal{M}^2 - 1)\rho\beta}{\mathcal{P}_0},$$

$$\frac{dT}{d\mathcal{M}} = (\gamma - 1)\beta\left[4\mathcal{M}_0\theta^3\theta' + (\gamma\mathcal{M}^2 - 1)r\right],$$

where $\beta(T, \mathcal{M})$ and $r(T, \mathcal{M})$ are known functions, and

$$\rho(T, \mathcal{M}) = \frac{\mathcal{M}_0}{\mathcal{M}\sqrt{T}}, \quad \theta(T, \mathcal{M}) = \frac{1}{\gamma\mathcal{P}_0}\left[K_m - 3\gamma\frac{\mathcal{M}_0^2}{\rho(T, \mathcal{M})} - 3T\rho(T, \mathcal{M})\right]$$

- Branch points no longer an issue.
- Nonsingular at $\mathcal{M} = 1$.

Outline

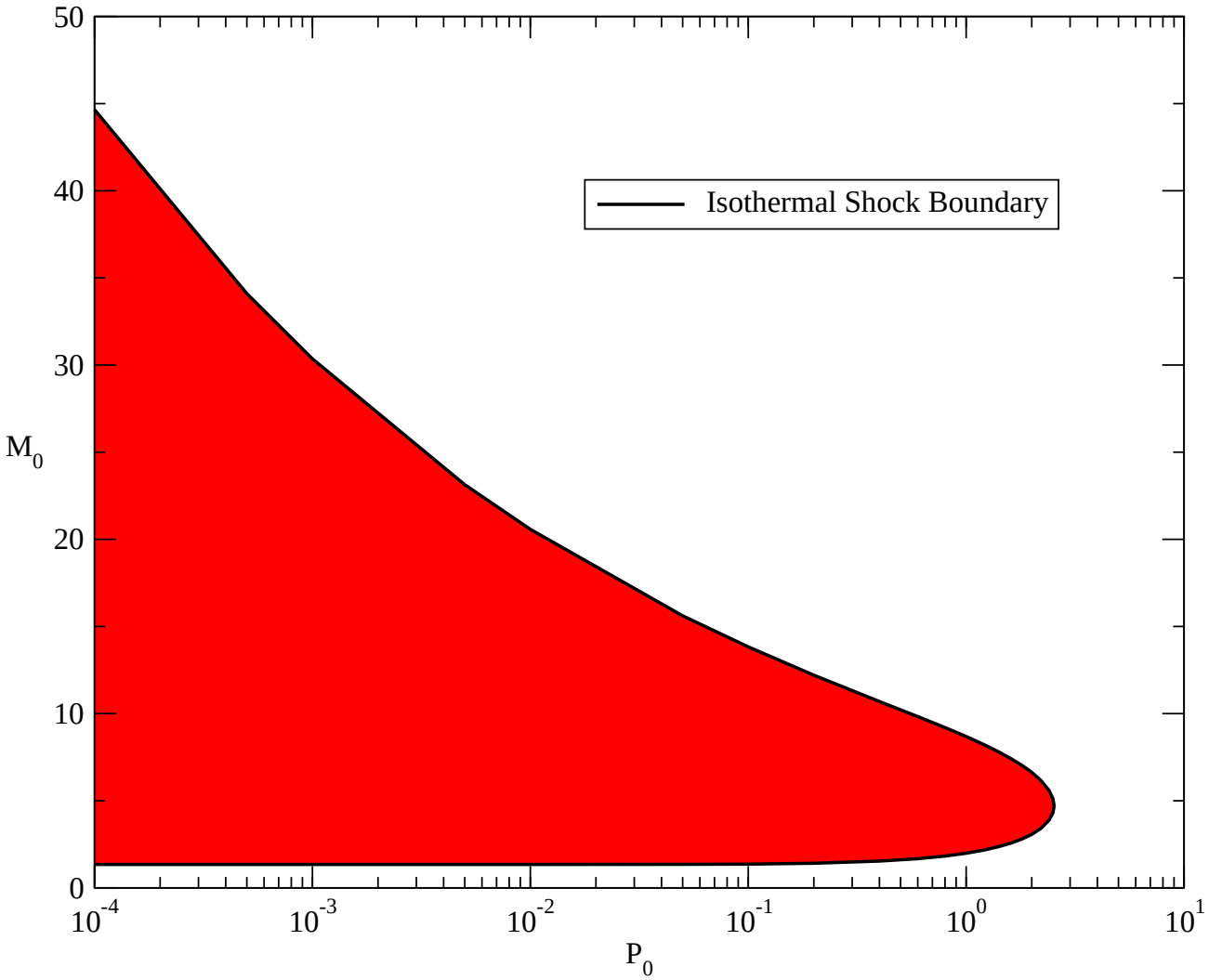
Solution Regimes

Solutions are characterized by the following:

- Is there an embedded hydro shock?
  - If no, all variables continuous.
- Is there a branch point ($\mathcal{M} = 1/\sqrt{\gamma}$)?

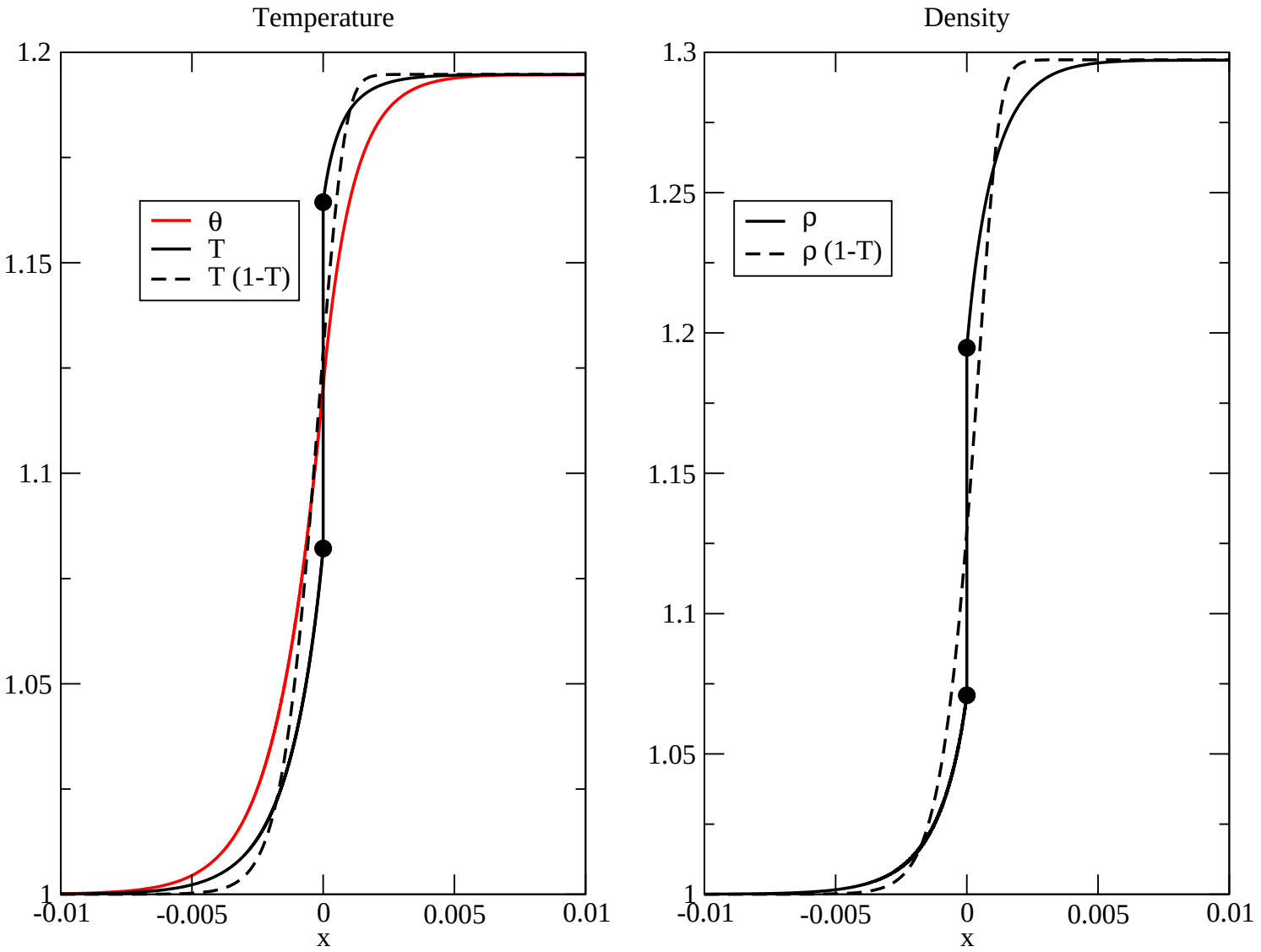
Isothermal Shock (1-T) $\Leftrightarrow$ Branch Point (2-T) Regions

$\gamma = 5/3$. See Lowrie & Rauenzahn (2007).



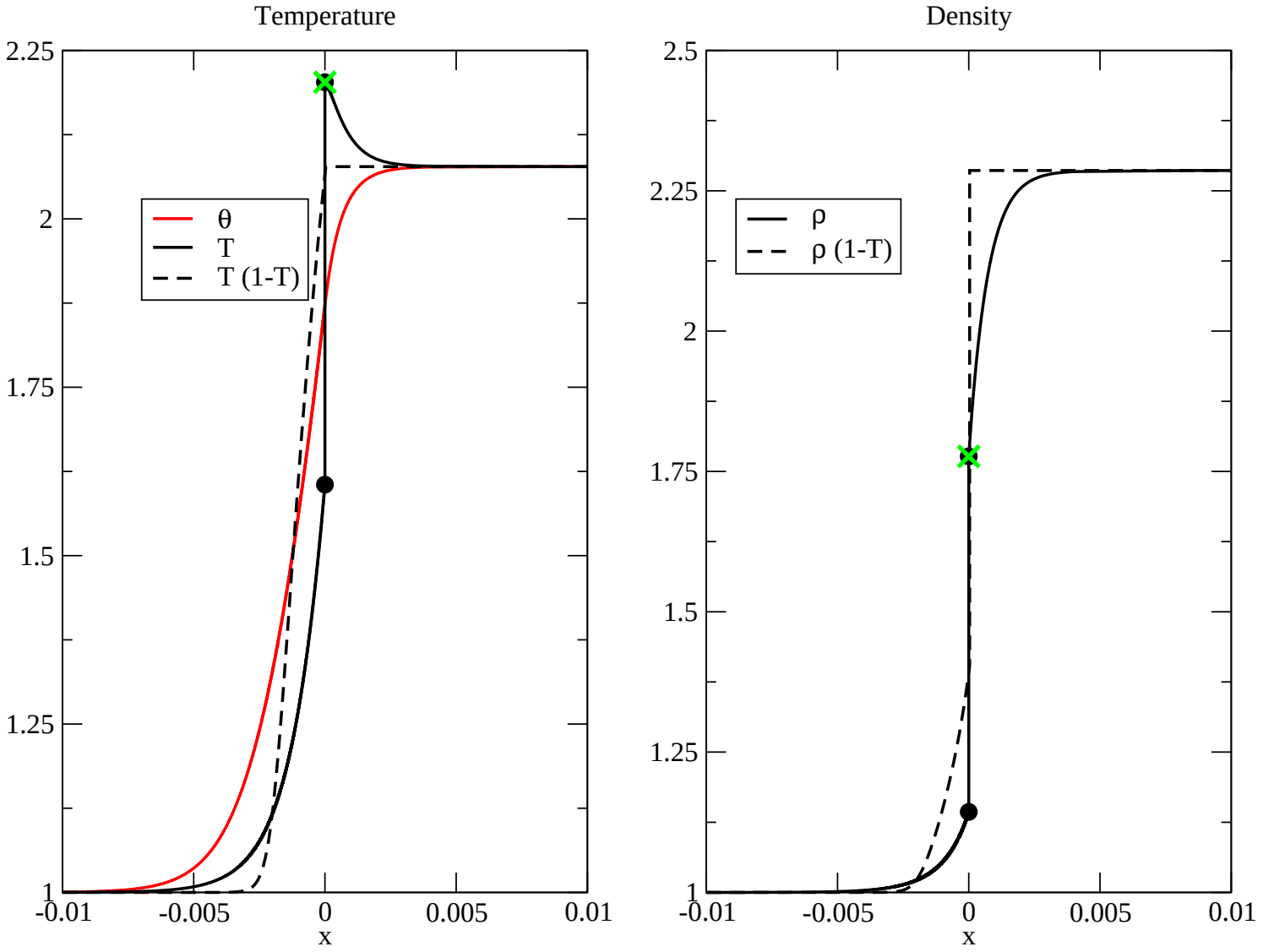
$\mathcal{M}_0 = 1.2$ solution

$\mathcal{P}_0 = 10^{-4}$, $\sigma_a = 10^6$, $\kappa = 1$, $\gamma = 5/3$; subcritical; shock, but no branch point.



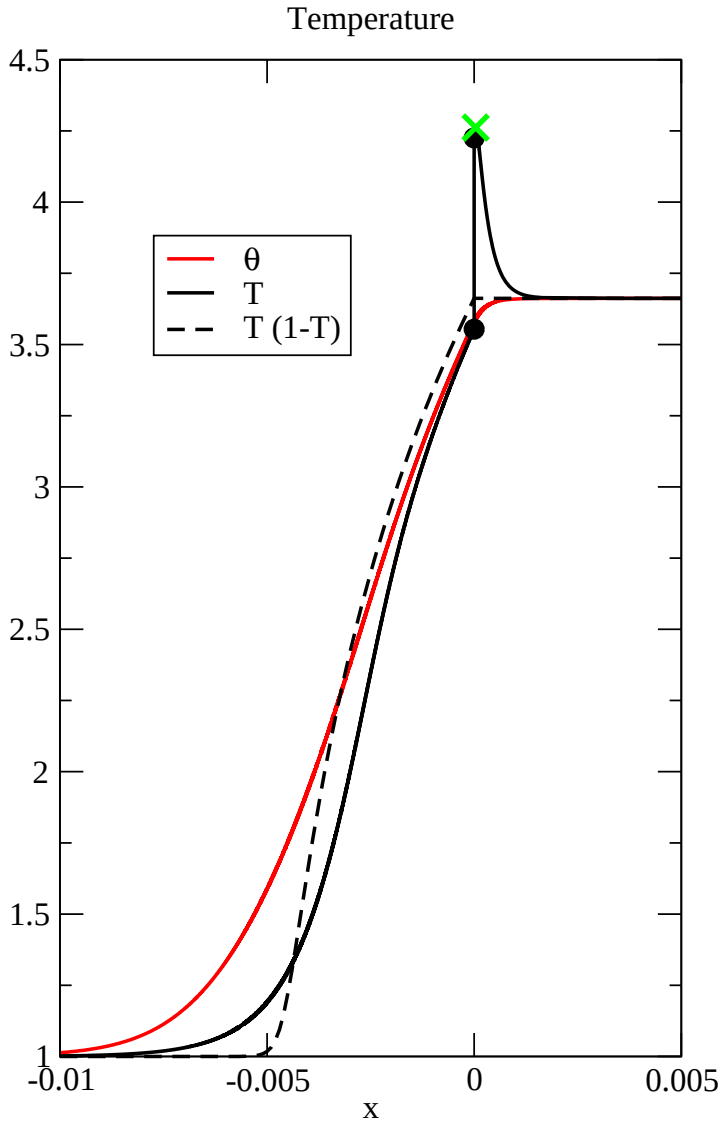
$\mathcal{M}_0 = 2$ solution

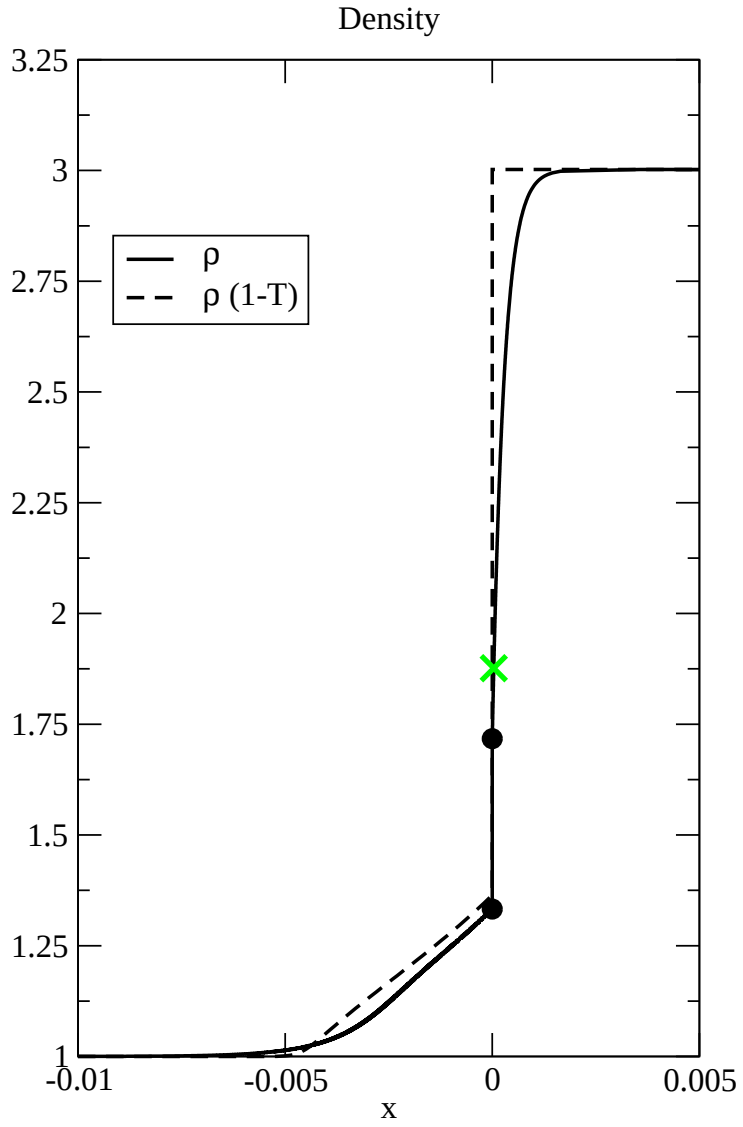
Subcritical; branch point coincident with shock.



$\mathcal{M}_0 = 3$ solution

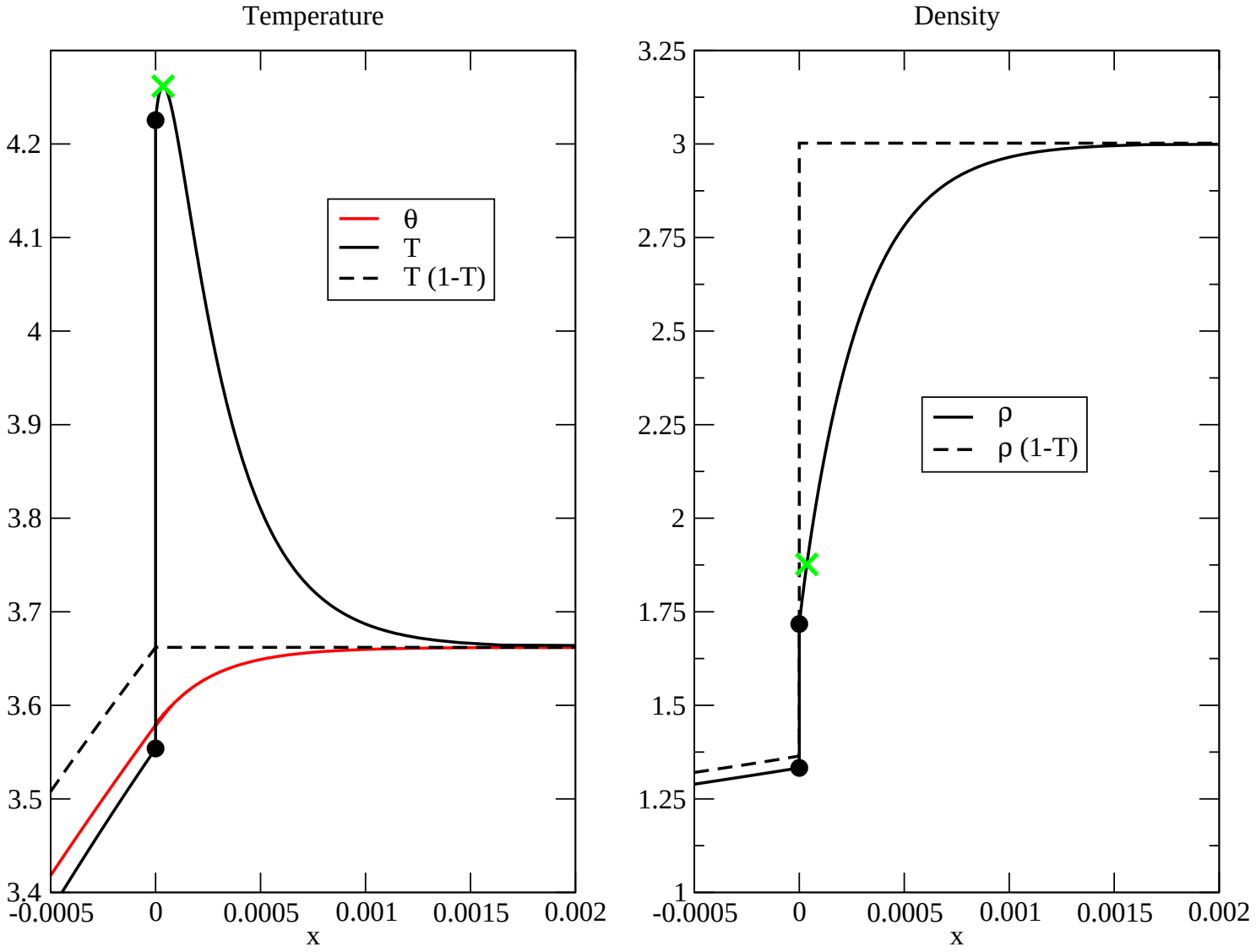
Subcritical; branch point downstream of shock.





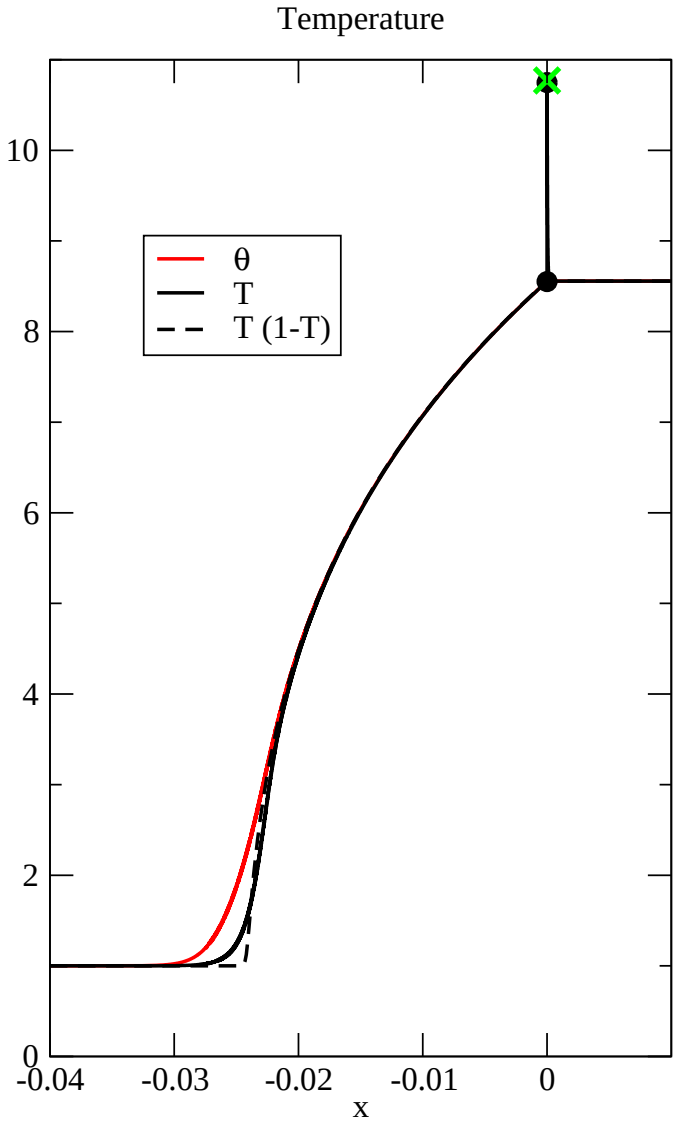
$\mathcal{M}_0 = 3$ solution

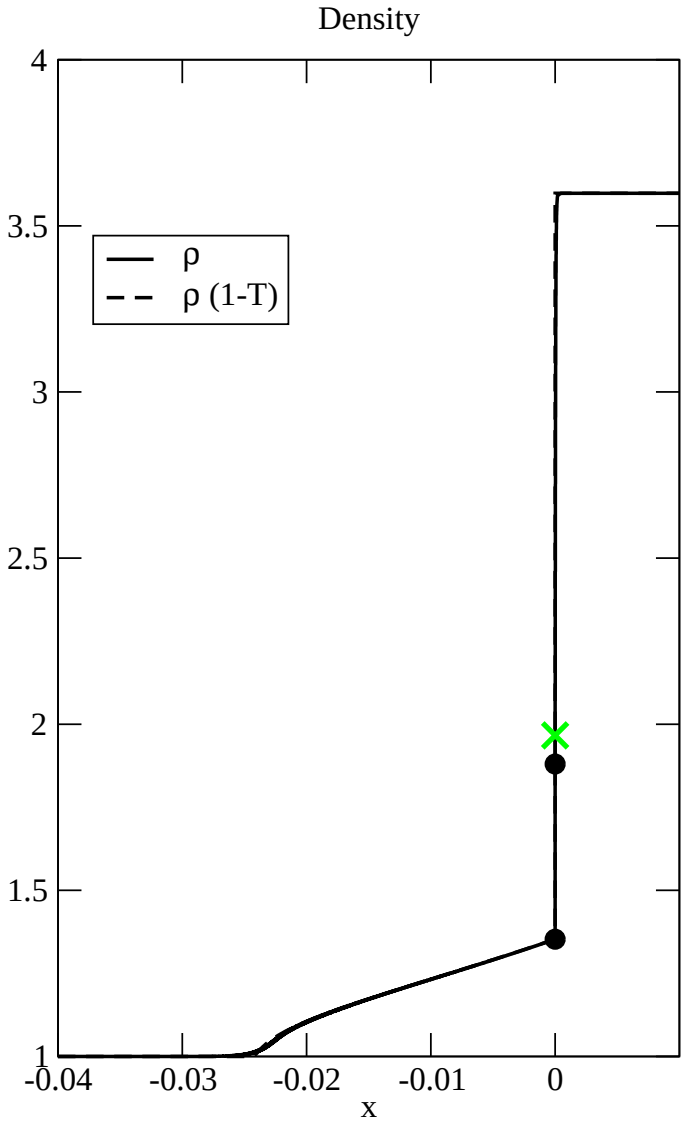
Spike region.



$\mathcal{M}_0 = 5$ solution

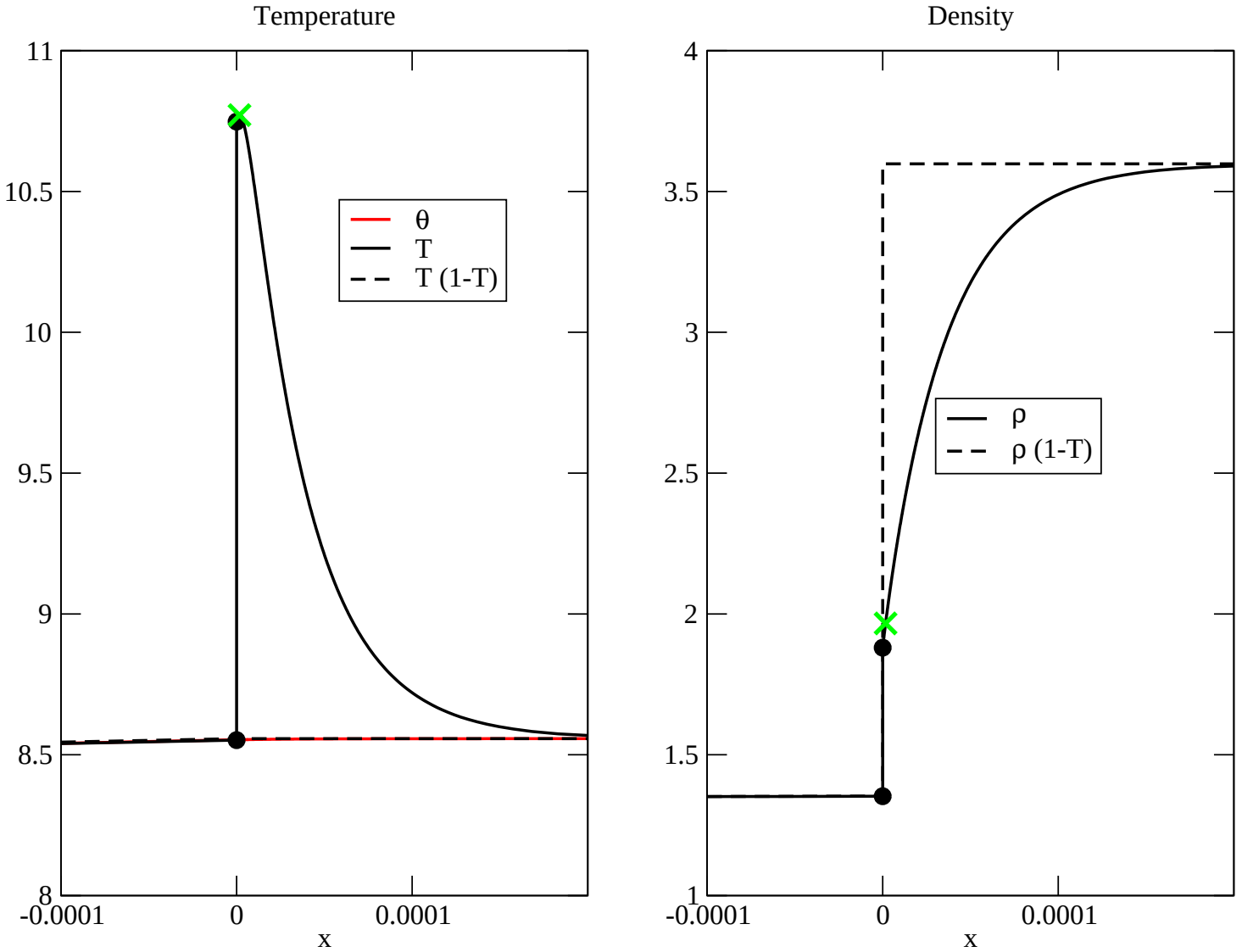
Supercritical; branch point downstream of shock.





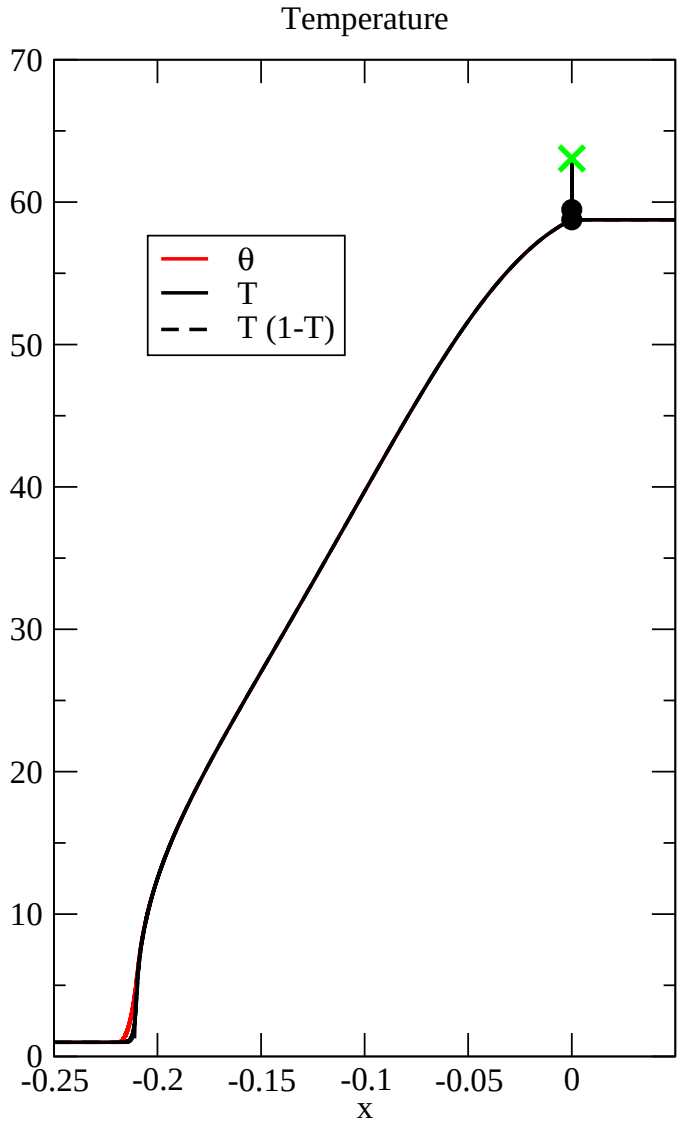
$\mathcal{M}_0 = 5$ solution (continued)

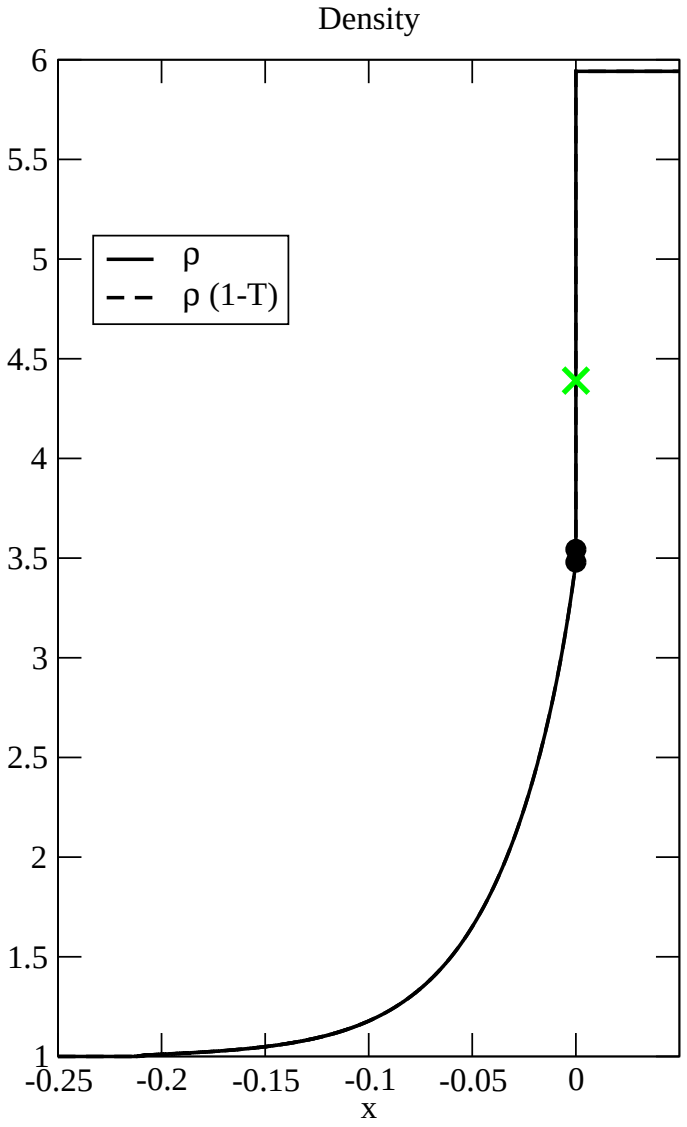
Zel'dovich spike region



$\mathcal{M}_0 = 27$ solution

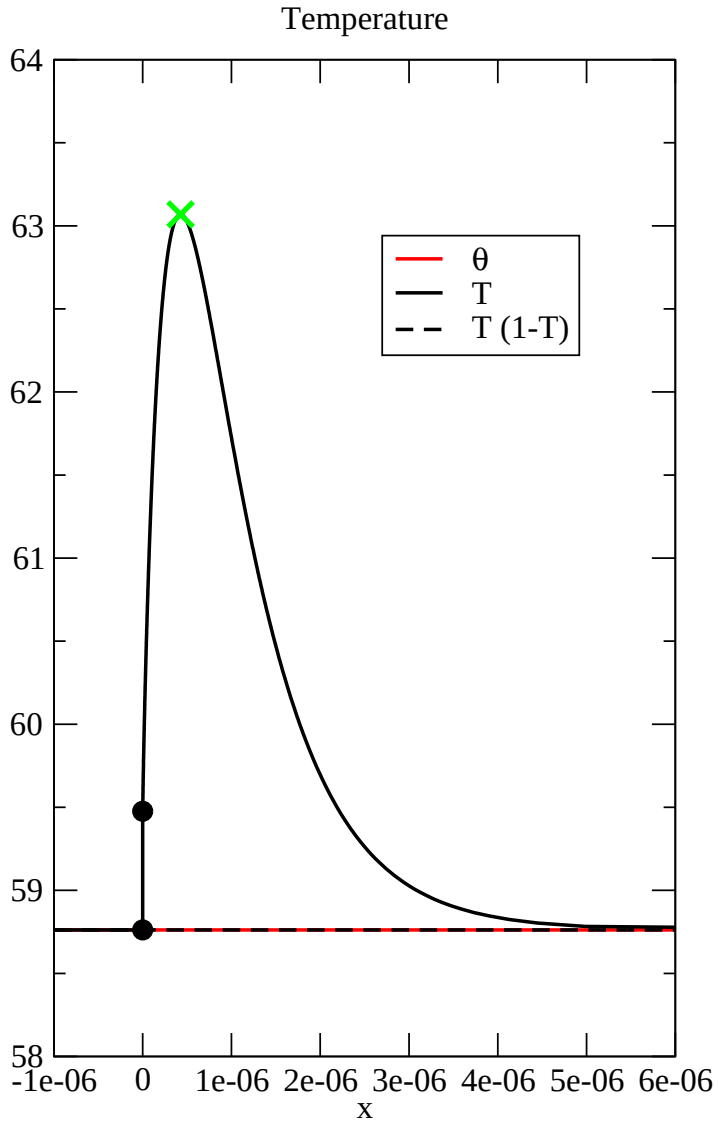
Supercritical; branch point downstream of shock. NOTE OVER COMPRESSION.

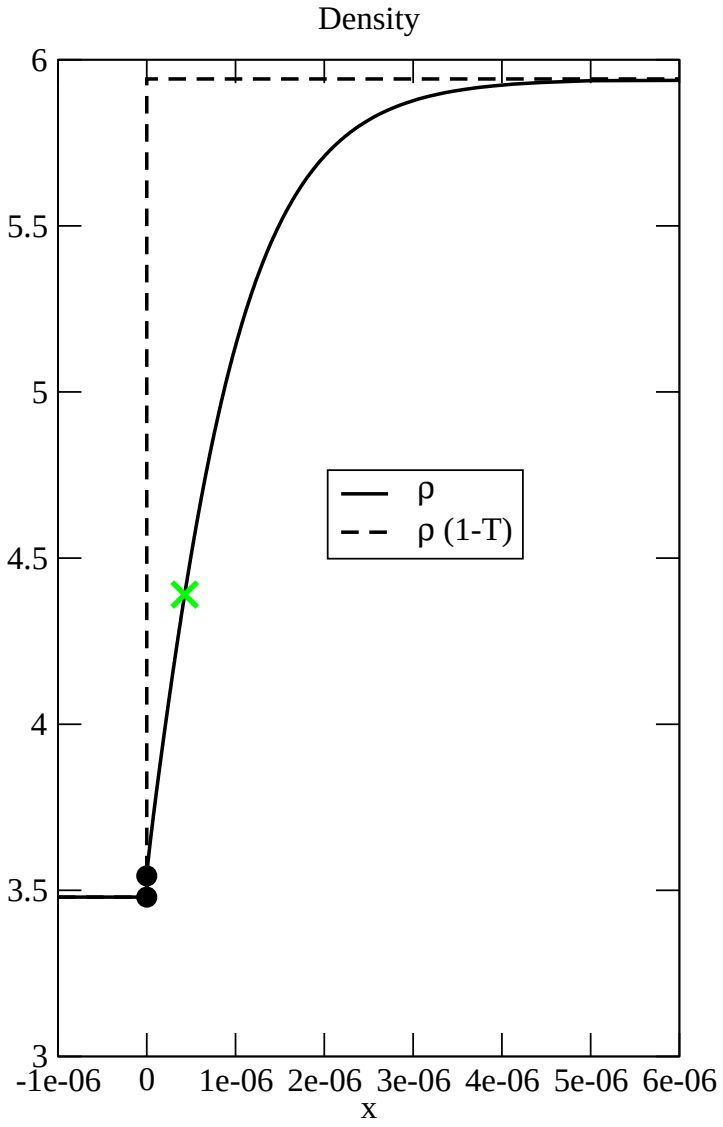




$\mathcal{M}_0 = 27$ solution (continued)

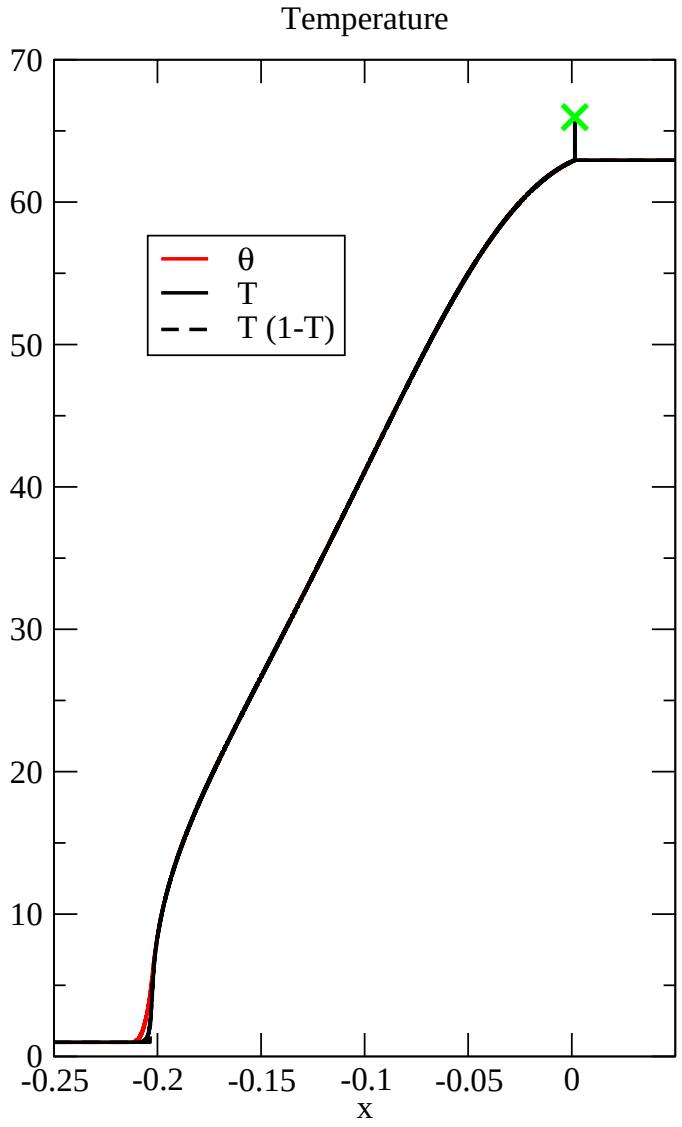
Zel'dovich spike region

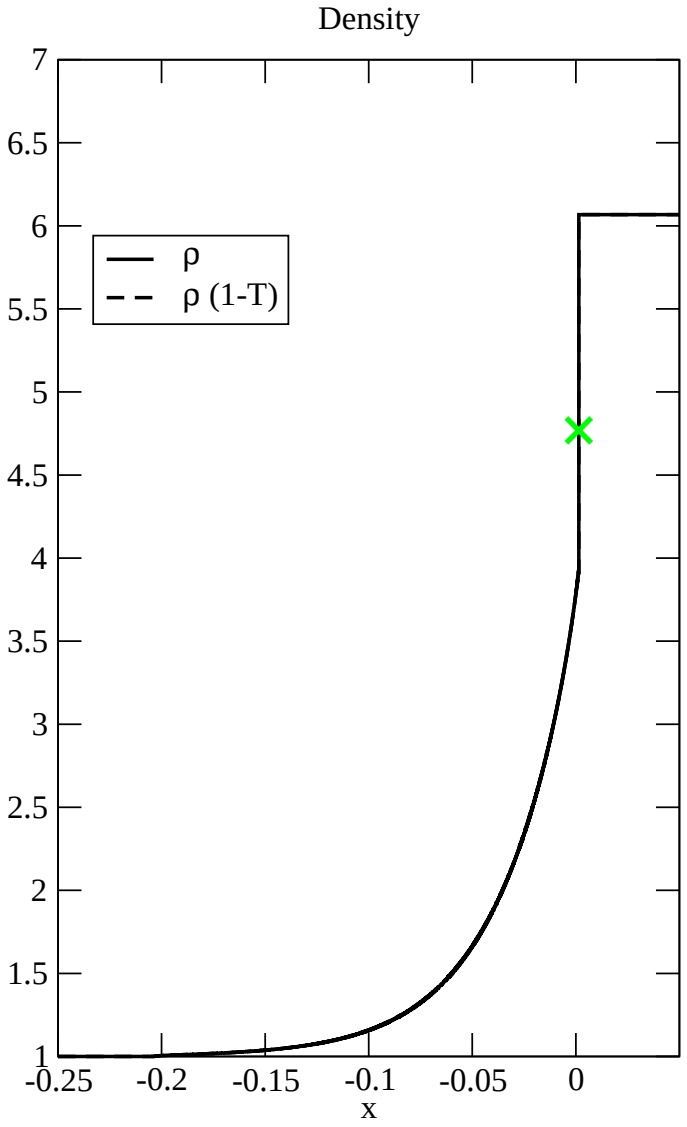




$\mathcal{M}_0 = 30$ solution

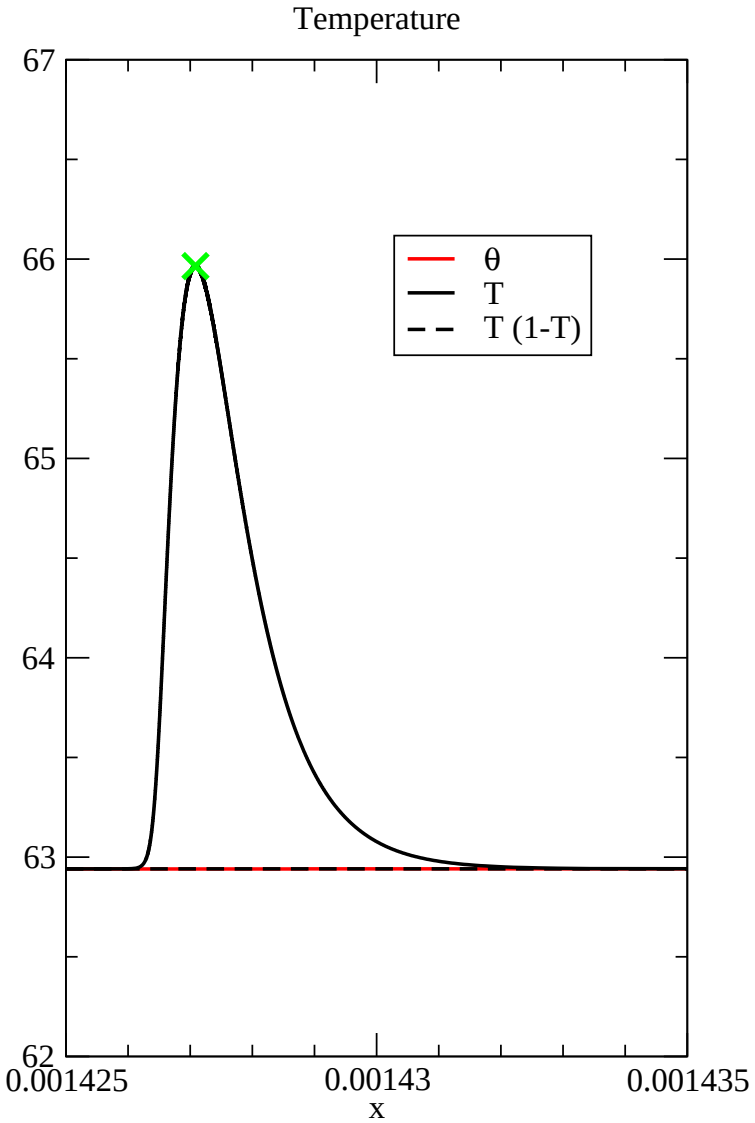
No shock, but still a branch point!

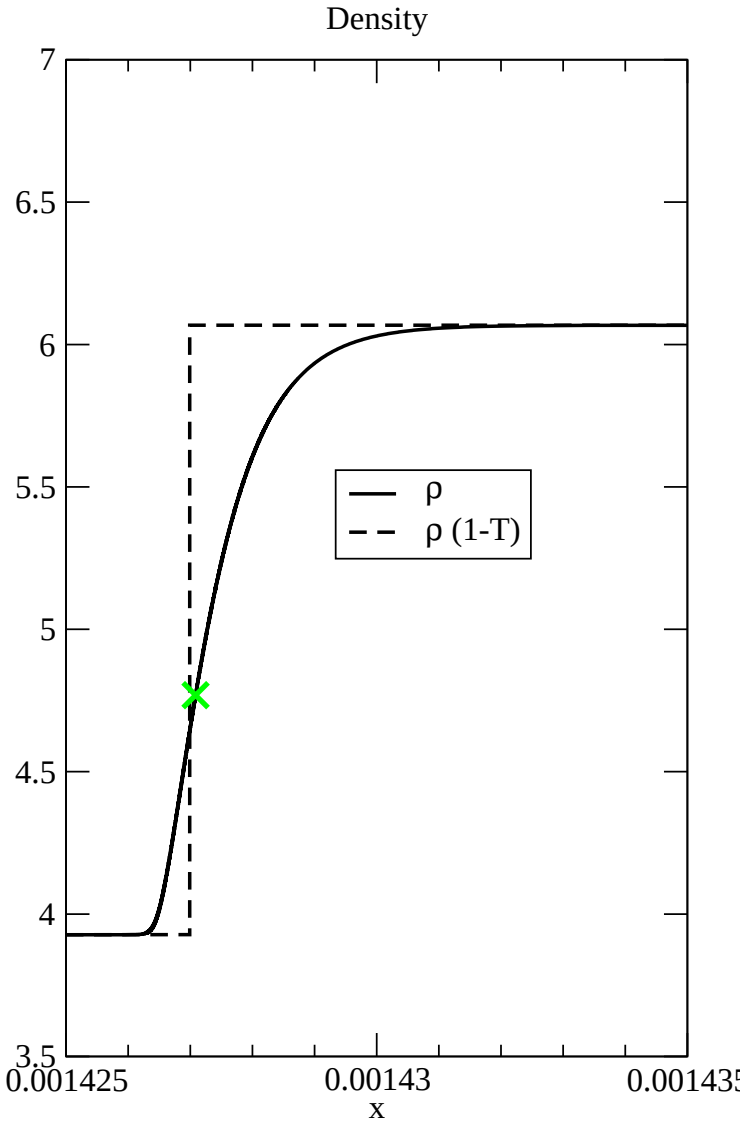




$\mathcal{M}_0 = 30$ solution (continued)

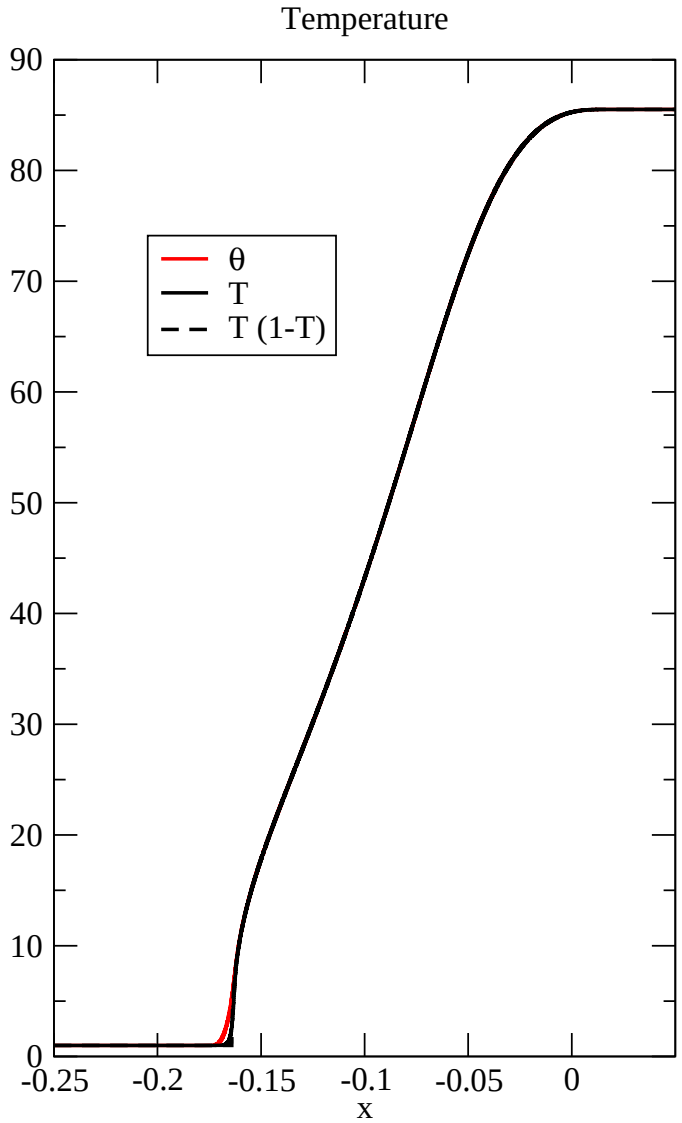
Spike region

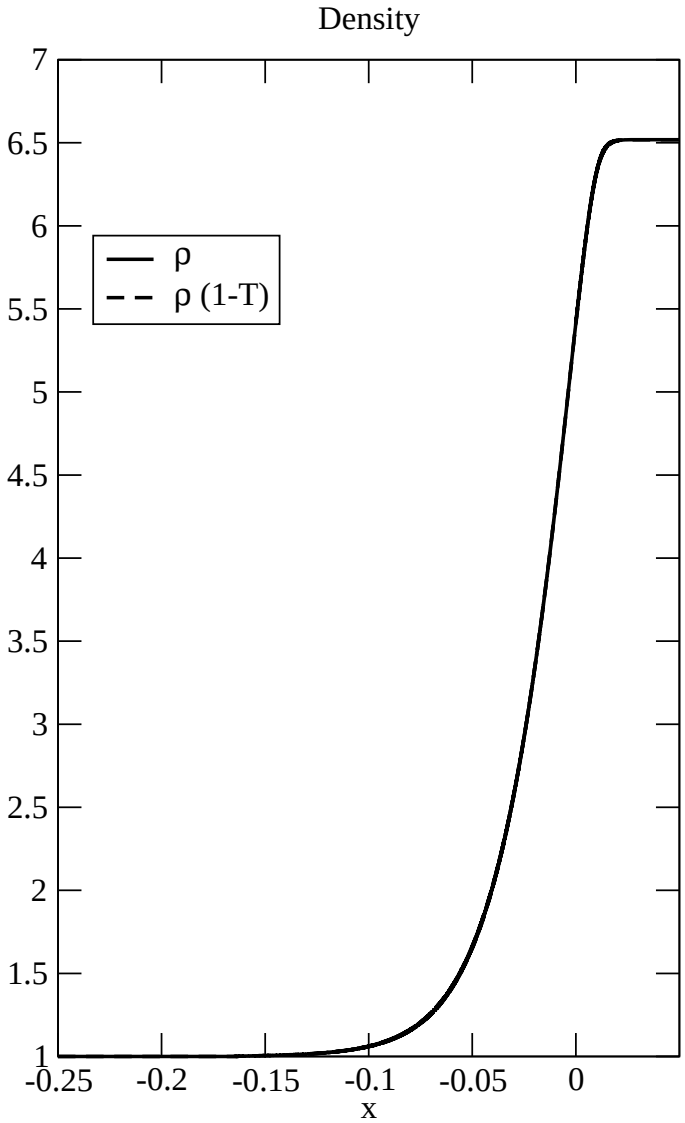




$\mathcal{M}_0 = 50$ solution

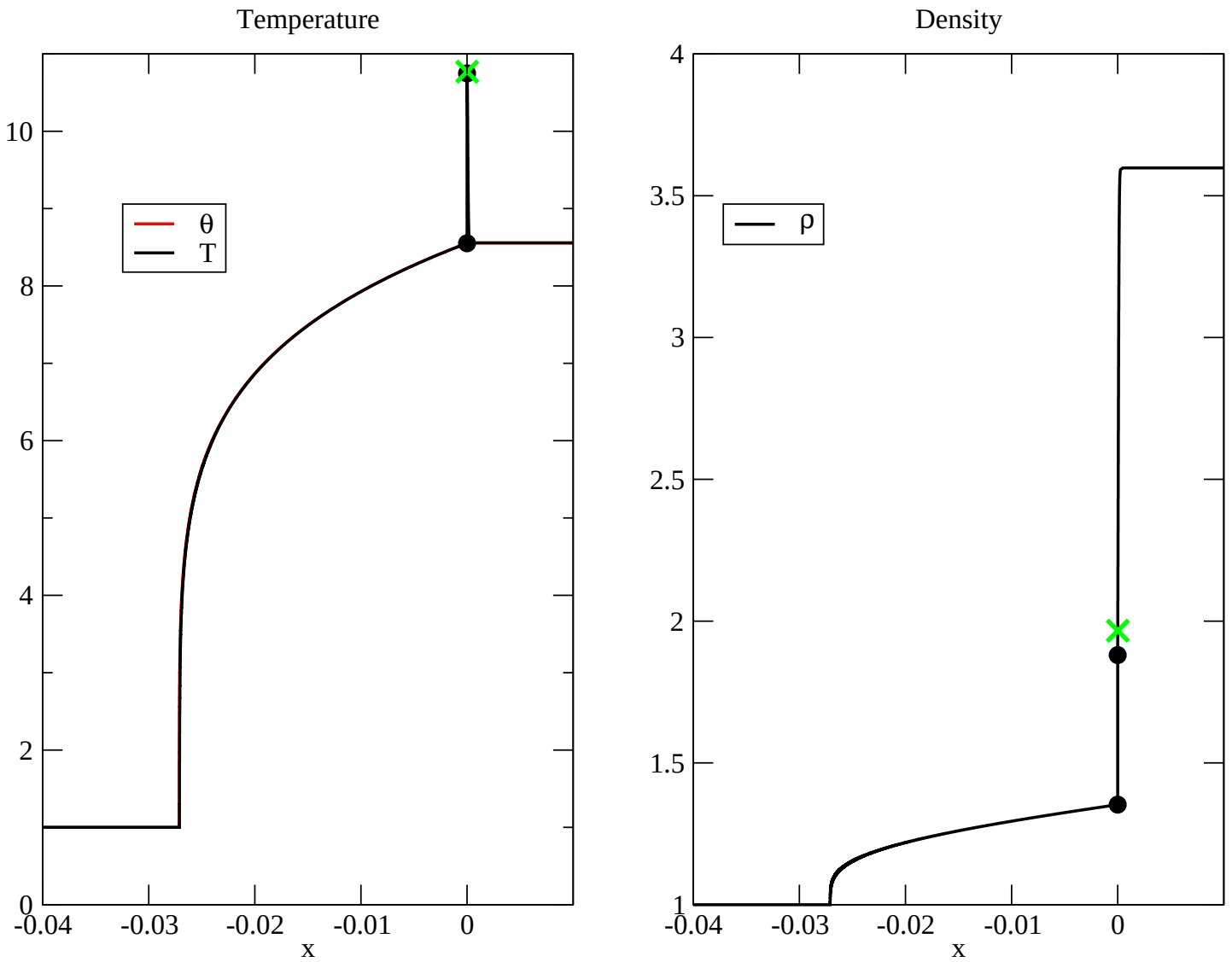
No shock or branch point





$\mathcal{M}_0 = 5$, variable cross-sections (Bremsstrahlung)

$\mathcal{P}_0 = 10^{-4}$, $\gamma = 5/3$, $\kappa = 0.00175 T^{7/2}/\rho$, $\sigma_a = 10^6/\kappa$



Outline

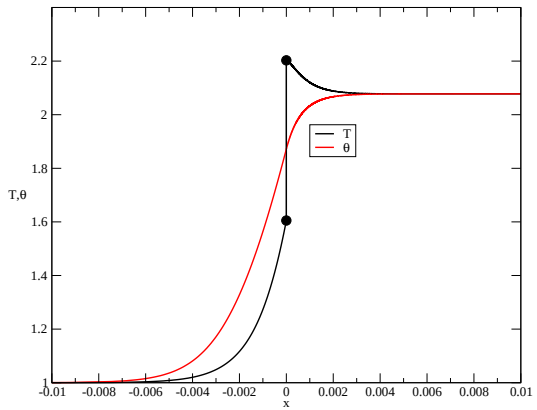
Branch Point at $T_{\max}$

- Assuming $\theta \approx$ constant in spike region, can prove that $T_{\max}$ occurs at the branch point ($\mathcal{M} = 1/\sqrt{\gamma}$).
- Previous work assumes T decreases monotonically from hydro shock. Ignores "$p\nabla \cdot \vec{v}$" contribution.

Estimates for $T_{\max}$

- Mihalas & Mihalas (1984) estimate:

$$T_{\max} = (3 - \gamma) T_1$$



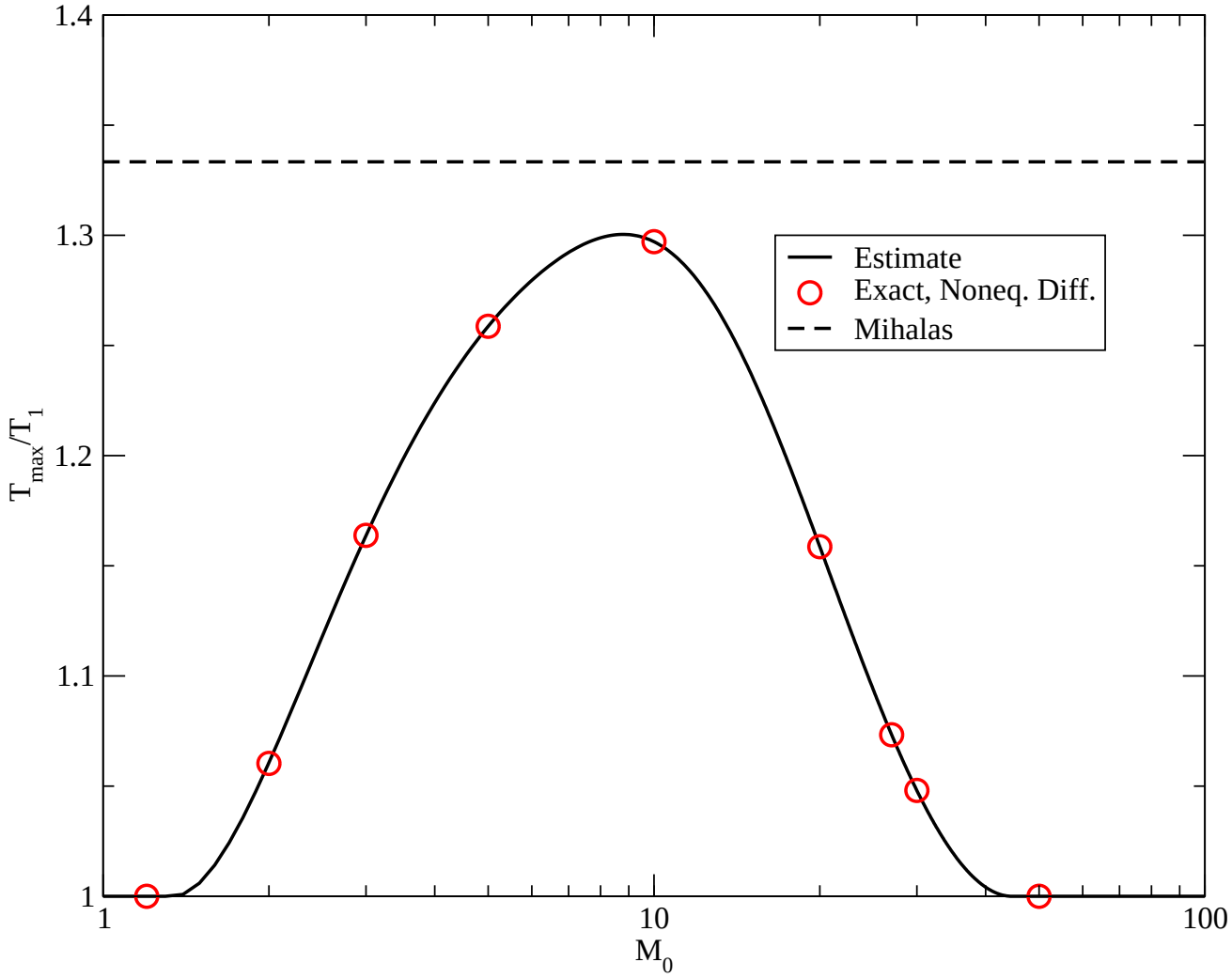
- Our new estimate:

$$T_{\max} = \begin{cases} \frac{1}{36\gamma\mathcal{M}_0^2}\left[3(\gamma\mathcal{M}_0^2 + 1) + \gamma\mathcal{P}_0(1 - T_1^4)\right]^2 & \text{if } \mathcal{M}_1 < 1/\sqrt{\gamma}, \\ T_1 & \text{otherwise.} \end{cases}$$

- Neither estimate requires detailed knowledge of profile; independent of cross-sections.

Comparison of $T_{\max}$ Estimates

$\mathcal{P}_0 = 10^{-4}$, $\gamma = 5/3$; exact also used $\sigma_a = 10^6$, $\kappa = 1$



Outline

Compare RAGE with Semi-Analytic Solution

Code verification example

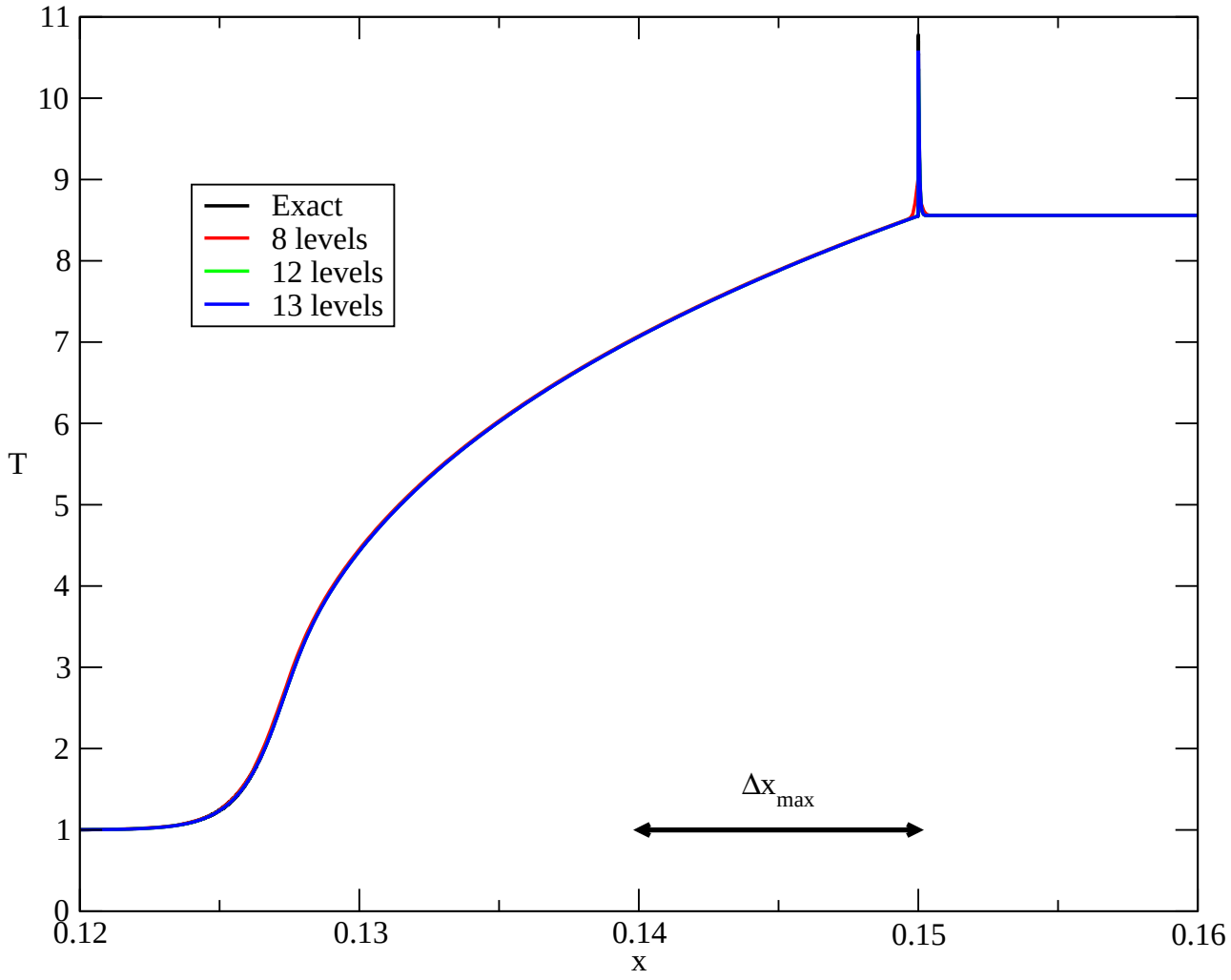
- RAGE code (Gittings *et al* 2006):
  - Cell-centered, high-resolution finite volume.
  - Operator split: Hydro explicit, radiation implicit
  - Uses adaptive mesh refinement (AMR)
- Run $\mathcal{M} = 5$ test case:
  - Initialize with exact shock profile
  - How well can RAGE propagate the shock and maintain profile?
  - Use AMR with

$$\frac{\Delta x_{\max}}{\Delta x_{\min}} = 2^{L-1}, \qquad L \in \{8, 10, 12, 13\}$$

For $L = 13$, used 967 cells, equivalent equally-spaced mesh is $204,800$ cells.

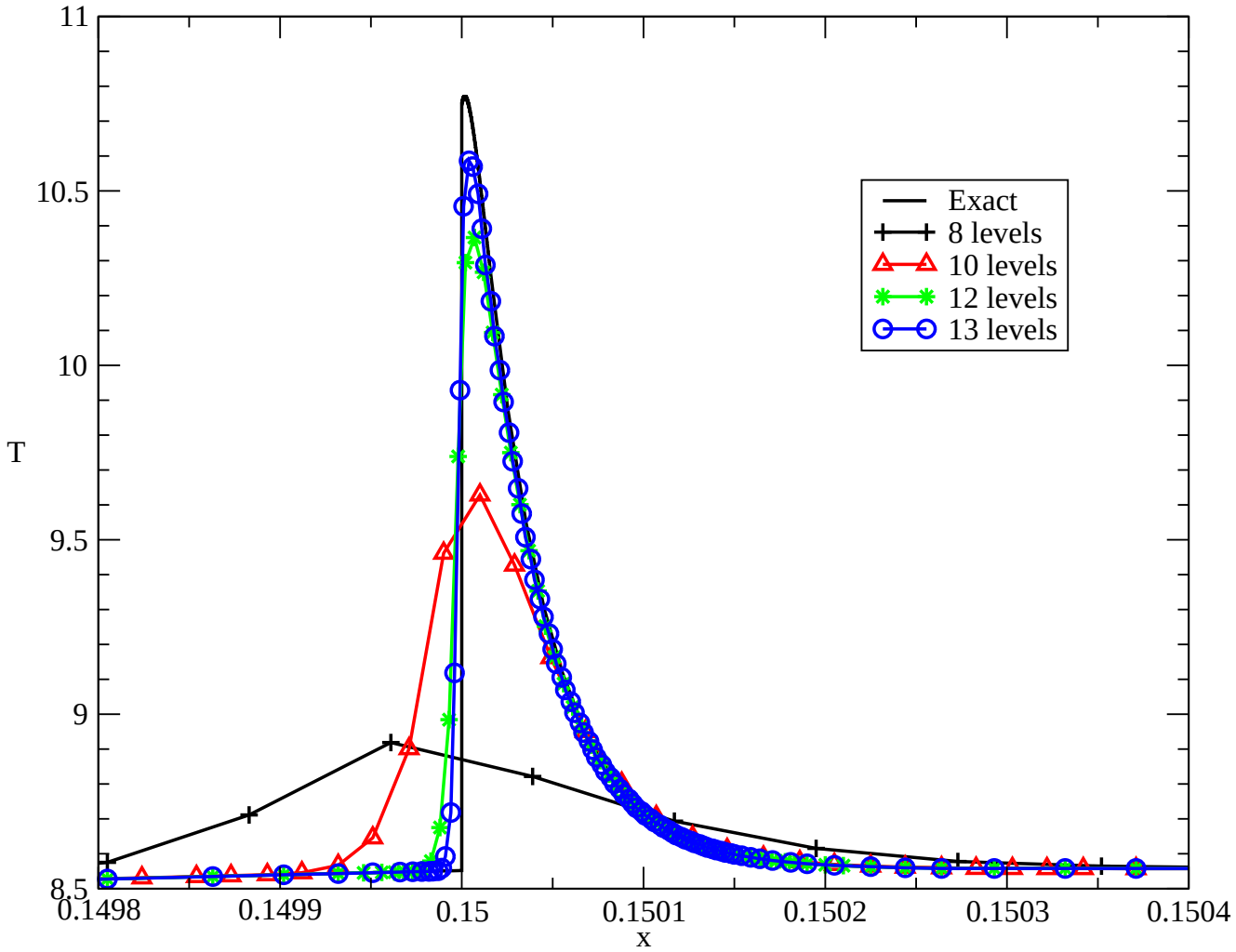
$\mathcal{M}_0 = 5$ RAGE Results

$\Delta x_{\text{max}} = 0.01$, shock propagated $\Delta x = 0.3$



$\mathcal{M}_0 = 5$ RAGE Results (continued)

Zel'dovich spike region



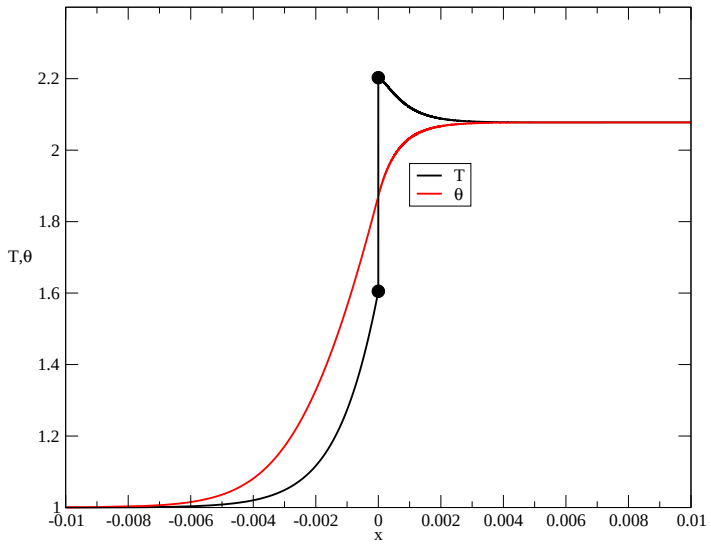
Summary

- Semi-analytic solutions for equilibrium and nonequilibrium, radiative shocks.
- Temperature may increase after hydro shock.
  - Classic picture: $T_{\max}$ occurs at hydro shock.
  - New picture: $T_{\max}$ occurs at branch point.
- Derived very accurate estimate for $T_{\max}$; independent of radiation model and cross sections.
- Solutions may be used for code verification.

- Future work
  - Separate ion/electron temperatures
  - Spherical symmetry
  - More advanced radiation models

Overview of Nonequilibrium Solution Procedure

1. Compute post-shock state ($x \to \infty$): Find root of a ninth-order polynomial in T_1.
2. Find pre-shock solution: Integrate ODEs from $\mathcal{M} = \mathcal{M}_0$ to $\mathcal{M} = 1$.
3. Find post-shock solution: Integrate ODEs from $\mathcal{M} = \mathcal{M}_1$ to $\mathcal{M} = 1$.
4. Connect the pre- and post-shock branches with a hydro shock.



Error in new $T_{\max}$ estimate

$\mathcal{P}_0 = 10^{-4}$, $\sigma_a = 10^6$, $\kappa = 1$, $\gamma = 5/3$