Second-Order Accurate Material-Order Independent Interface Reconstruction for Multi-Material Flows

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- Interface reconstruction is well-developed for two-material interfaces

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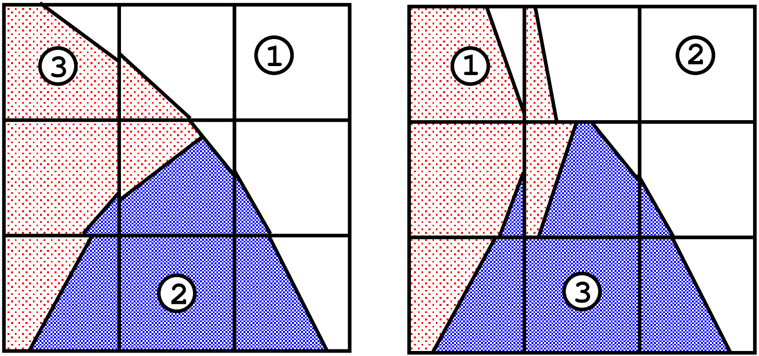
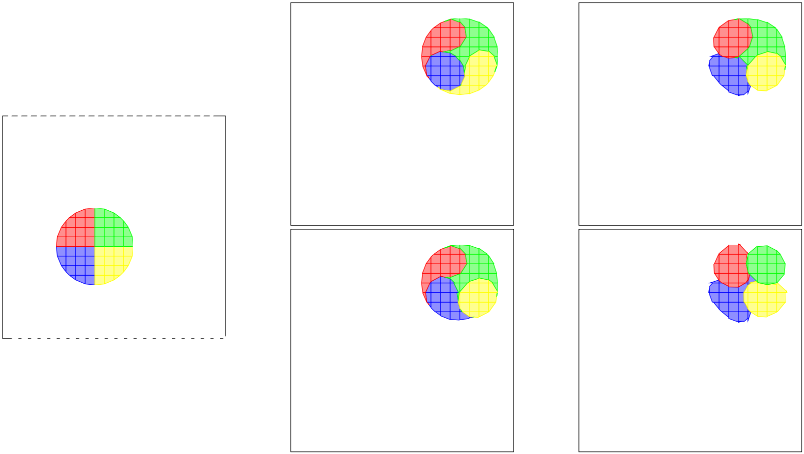
- Aggravates the problem of “flotsam” and “jetsam”

Interface Reconstruction using Different Material Orderings



(a) (b)

Order Dependence in Advection of Multi-Material Disc



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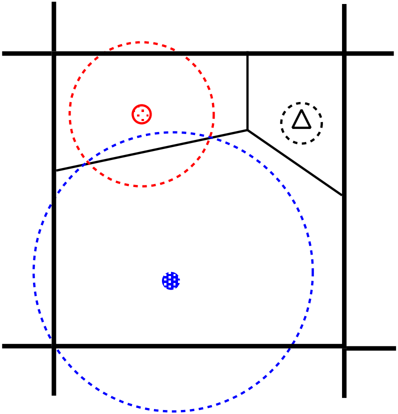

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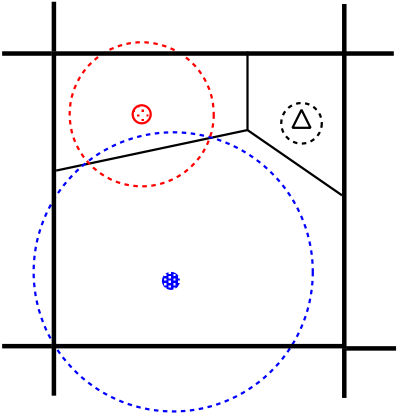
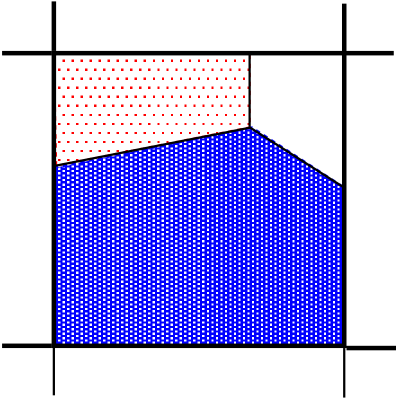


(c)

Subdivide cell using a weighted Voronoi (power) diagram



(d)

Material Location

Material Location Strategies

- Particle attraction-repulsion [Garimella, et. al.]
  Can capture filaments but slow, unstable,
- Low order quadrature [Mosso, et. al.]
  Fast, not very accurate
- Linear reconstruction [Garimella et. al.]
  Fast, accuracy $\approx$ Gradient-based method
- Higher order reconstructions, cellular automata
  Under investigation

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- Approximate material centroid in the cell is computed by

$$\bar{\mathbf{x}}_i^m = \frac{\int_\Omega \xi^m(\mathbf{x})\mathbf{x}d\mathbf{x}}{\int_\Omega \xi^m(\mathbf{x})d\mathbf{x}} = \frac{\int_\Omega \xi^m(\mathbf{x})\mathbf{x}d\mathbf{x}}{\|\Omega\| f_i^m}$$

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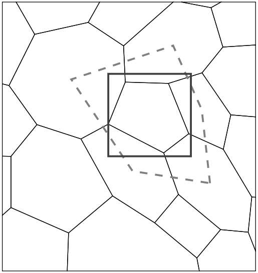
- Use a square around cell with the same center as the cell center
- For 2 materials, identical to gradient-based reconstruction



Dark line: Path for centroid
Dashed line: Path for gradient

Power Diagrams

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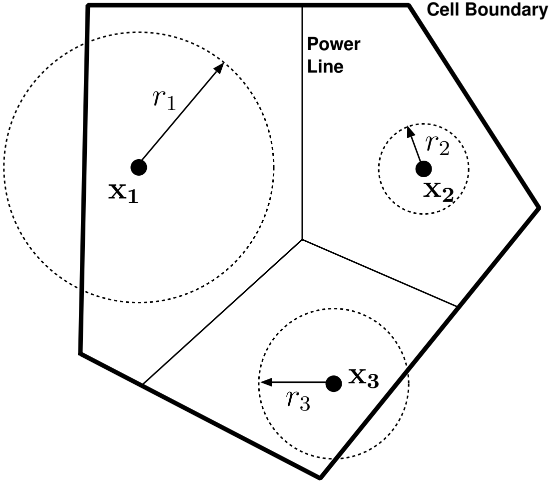
Power Diagram is the union of power cells for all sites

Truncated Power Diagram in Cell



Properties of Power Diagrams

- A power line of two sites is perpendicular to the line connecting the two sites
- Power lines for three non-collinear sites intersect at a point
- Power cells are convex
- The site corresponding to a power cell may lie outside the cell
- Like Voronoi diagrams, power diagrams partition a space into convex polyhedra (some of which may be open)
- Power diagrams are the closest analogues of Voronoi diagrams

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Special treatment of overshoots (i.e. when weights generate a power diagram subcell outside the mesh cell)

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- Procedure is second-order accurate and exactly recovers straight lines

Smoothing objective function

Primary objective: Minimize the C^0 and C^1 discontinuity between interface segments in neighboring cells

Quantified as discrepancy between normal of interface segment and normal of segment connecting midpoints of interface segment and neighbor (Swartz)

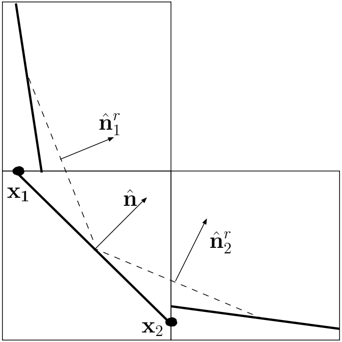
$\phi = \sum_i \hat{\mathbf{n}} - \hat{\mathbf{n}}_i^r$

Constraints:

Match Volume fraction exactly

Maintain convexity of cells

Respect cell boundaries



Summary of Method

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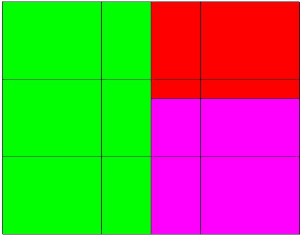
Results

Simple Example - Three Materials, Structured Grid

Gradient method

Ordering 0



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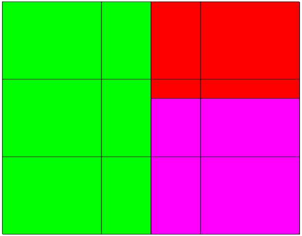

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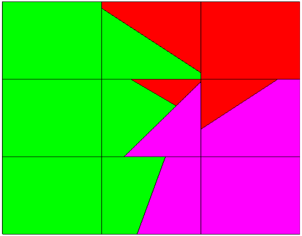

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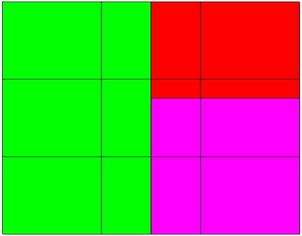

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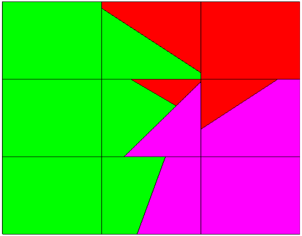
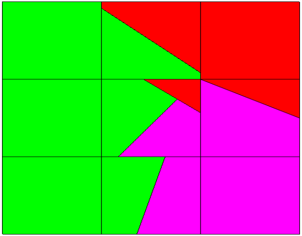


Gradient method
Ordering 2



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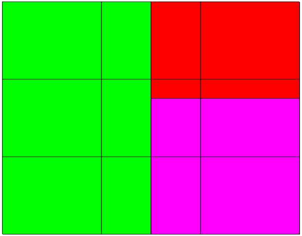

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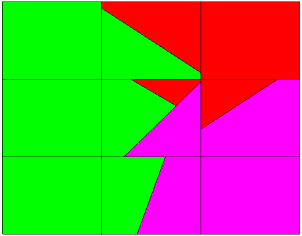

Gradient method
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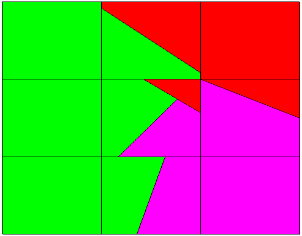
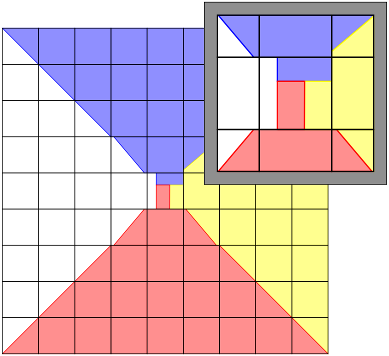
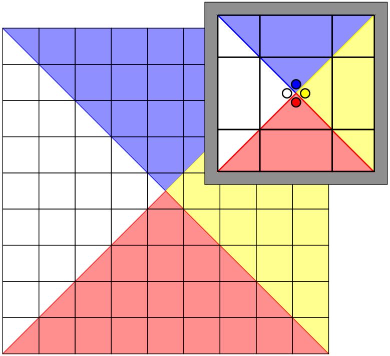


Power Diagram method
Order Independent !

Four Materials, Structured Grid

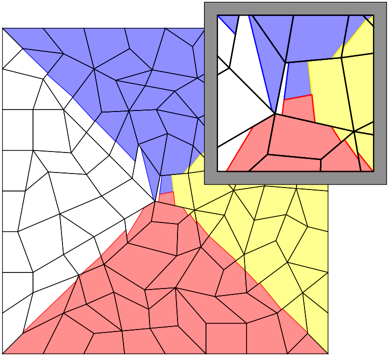


Gradient method

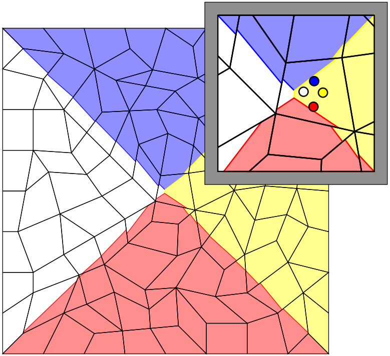


New Method

Four Materials, Untructured Grid

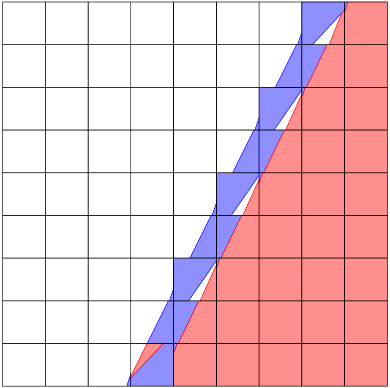


Gradient method
incorrect ordering



New method
order-independent

Three-Material Filament



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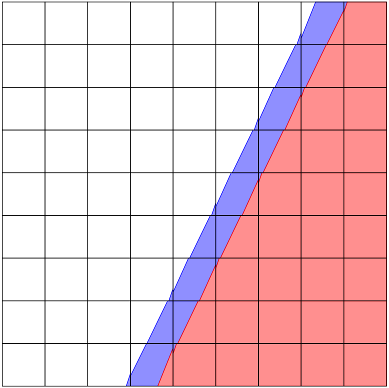
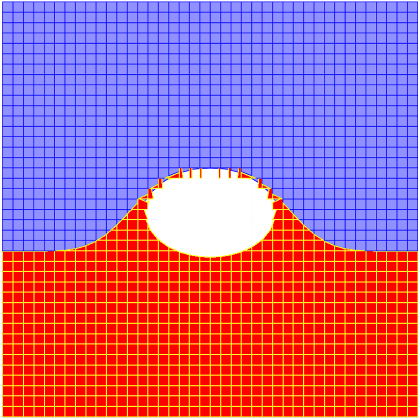


New method
Order-independent

Gas Bubble Rising to the Surface



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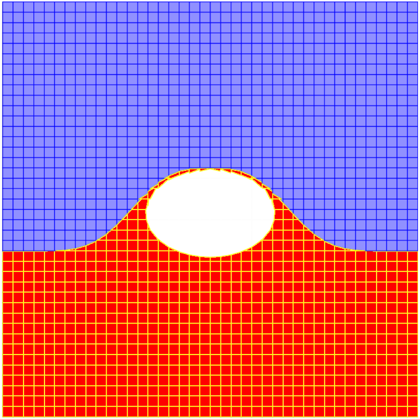

New method
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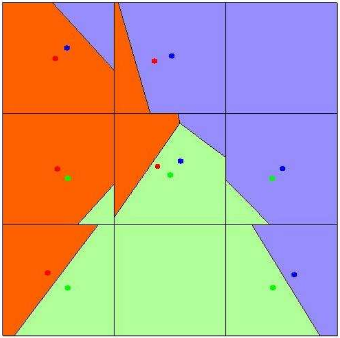
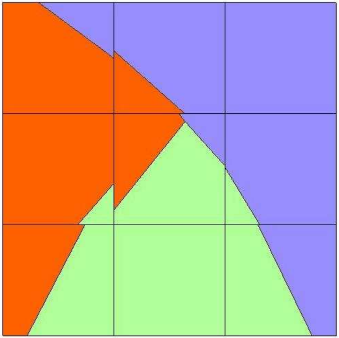
Multi-material Interface Smoothing - Three Material Example

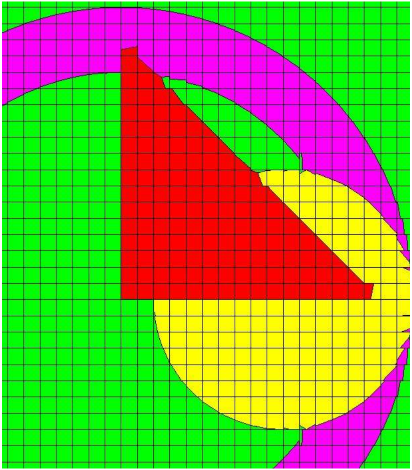


Without Smoothing



With Smoothing

Multi-material Interface Smoothing - Bailey Example



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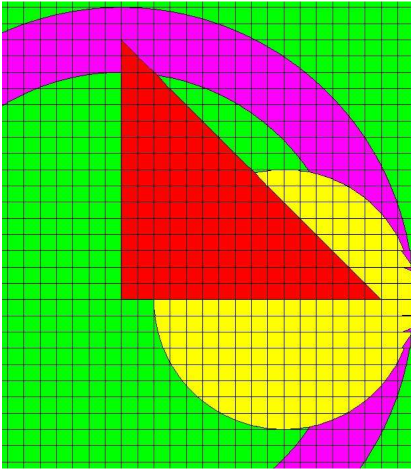
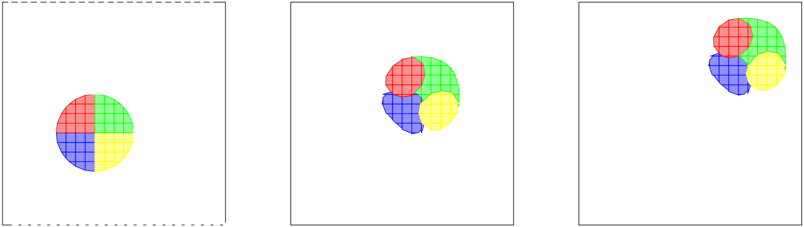


With Smoothing

Advection of multi-material bubble (Gradient Method)

Material Order dependent Gradient-based Algorithm

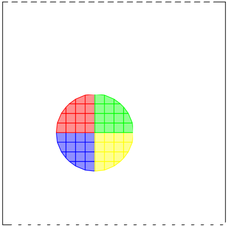


Initial configuration — After 50 timesteps — After 100 timesteps

40x40 grid, Diagonal Movement with velocity of (1.1,1.1)

Advection of multi-material bubble (New Method)

New Material Order Independent Method (No Smoothing)



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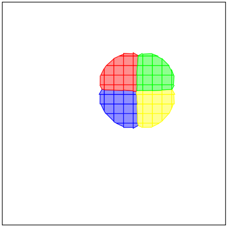
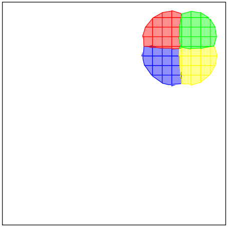
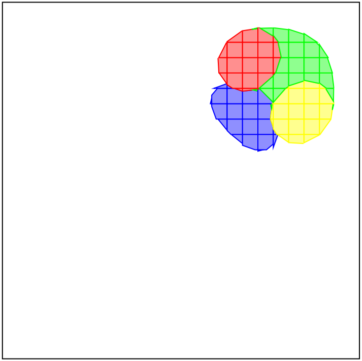


After 50 timesteps

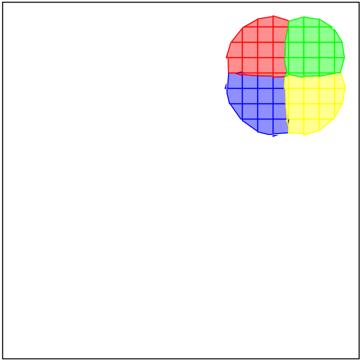


After 100 timesteps

Multi-material Bubble Advection - Comparison of Final States



Gradient Method



New Method

Convergence

- Measure error in terms of symmetric area difference
  $\Delta = |\Omega_1| + |\Omega_2| - 2|\Omega_1 \cap \Omega_2|$,
  Ω_1: Coarse scale reconstruction,
  Ω_2: "Exact" or finest scale reconstruction
- 2nd order accurate methods for smooth two material interfaces maintain order of accuracy in the presence of multi-material junction
- This includes LVIRA, Swartz and our method
- However, wrong ordering in a multi-material filament will drop accuracy of traditional methods to first order
- We maintain second order accuracy

Conclusions

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- Can be generalized easily to 3D

Future Work

- Integrate smoothing into advection tests
- Integrate into real hydrocodes
- Explore more accurate centroid approximation methods