A 3D Finite Volume Lagrangian Scheme

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Euler equations in Lagrangian coordinates

Lagrangian compressible gas dynamics: $\tau = \frac{1}{\rho}$, $p = (\gamma - 1)\frac{\varepsilon}{\rho}$

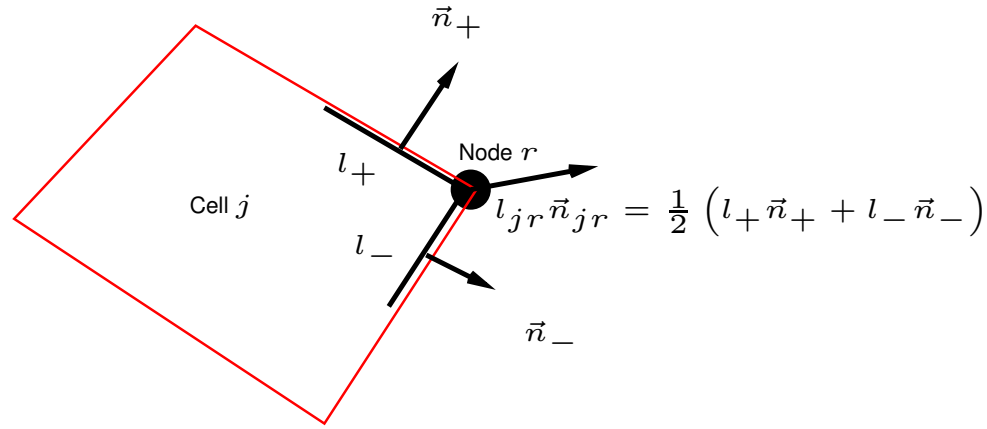
$$\begin{cases} \rho d_t \tau - \nabla \cdot \mathbf{u} = 0, \\ \rho d_t \mathbf{u} + \nabla p = 0, \\ \rho d_t e + \nabla \cdot (p\mathbf{u}) = 0. \end{cases}$$

- Lagrangian fluid dynamics is critical for ICF.
- Finite Volume cell centered schemes have a great potential for **ALE** developments and **remeshing** techniques since

  $$\rho, \mathbf{u} \text{ and } e = \varepsilon + \frac{1}{2}|u|^2$$

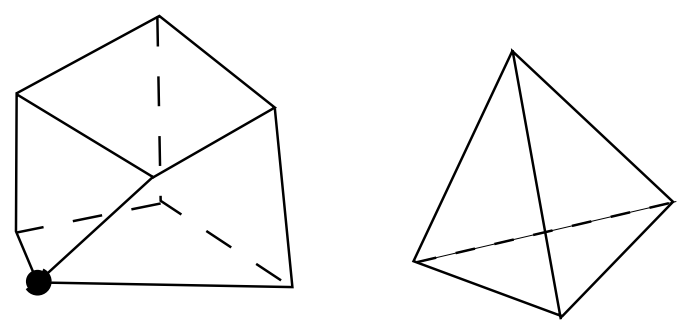
  are discretized at the same place: the density ρ_j^n, the velocity $\mathbf{u}_j^n$ and the **total energy** e_j^n are the unknowns given in cell j.
- Presentation of ongoing research about the 3D extension.

Problem adressed



The Lagrangian mass is $M_j = s_j(t)\rho_j(t)$. Define $\mathbf{C}_{jr} = l_{jr}\vec{n}_{jr} \in \mathbb{R}^2$. Then

$$s_j = \frac{1}{2}\sum_r \left(\mathbf{C}_{jr}, \mathbf{x}_r\right), \quad s_j'(t) = \sum_r \left(\mathbf{C}_{jr}, \mathbf{u}_r\right), \text{ and } \sum_j \mathbf{C}_{jr} = 0.$$



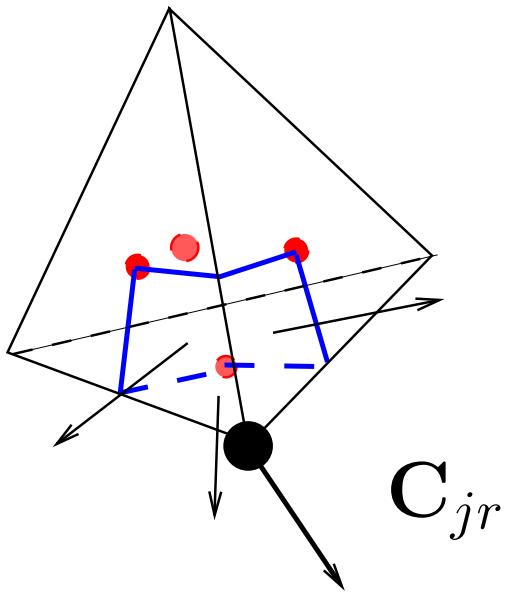
The idea is to define $\mathbf{C}_{jr} \in \mathbb{R}^3$ in 3D for general cells (on the left).

$\mathbf{C}_{jr}$ in 3D

One can construct the vectors $\mathbf{C}_{jr} \in \mathbb{R}^3$ such that the volume of cell j is

$$V_j = \frac{1}{3} \sum_r (\mathbf{C}_{jr}, \mathbf{x}_r), \quad V_j'(t) = \sum_r (\mathbf{C}_{jr}, \mathbf{u}_r), \quad \mathbf{u}_r(t) = \mathbf{x}_r'(t).$$

For tetrahedra $\mathbf{C}_{jr}$ are *uniquely* defined.



For other cell types $\mathbf{C}_{jr}$ are ***not*** *uniquely* defined.

In any case, one has $\sum_j \mathbf{C}_{jr} = 0$.

Discretization of the divergence

Define the mass of the lagrangian cell

$$M_j = \rho_j(t) V_j(t).$$

One has

$$M_j \frac{d}{dt}\left(\frac{1}{\rho_j(t)}\right) = V_j'(t) = \sum_r (\mathbf{C}_{jr}, \mathbf{u}_r)\,.$$

Therefore

$$\sum_r (\mathbf{C}_{jr}, \mathbf{u}_r) \approx \int_{\partial j} (\mathbf{u}, \mathbf{n}) d\sigma \qquad \left(= \int_j \nabla \cdot \mathbf{u}\, dx\right)$$

is our Finite Volume discretization of the divergence operator.

$\Longrightarrow$ The $\mathbf{C}_{jr}$'s give the discretization of the divergence operator.

Discretization of the gradient

We use the idea of a **compatible discretization** of all discrete operators. One has the formula

$$\sum_r \mathbf{C}_{jr} p_r \approx \int_{\partial j} \mathbf{n} p \, d\sigma \qquad \left(= \int_j \nabla p \, dx \right).$$

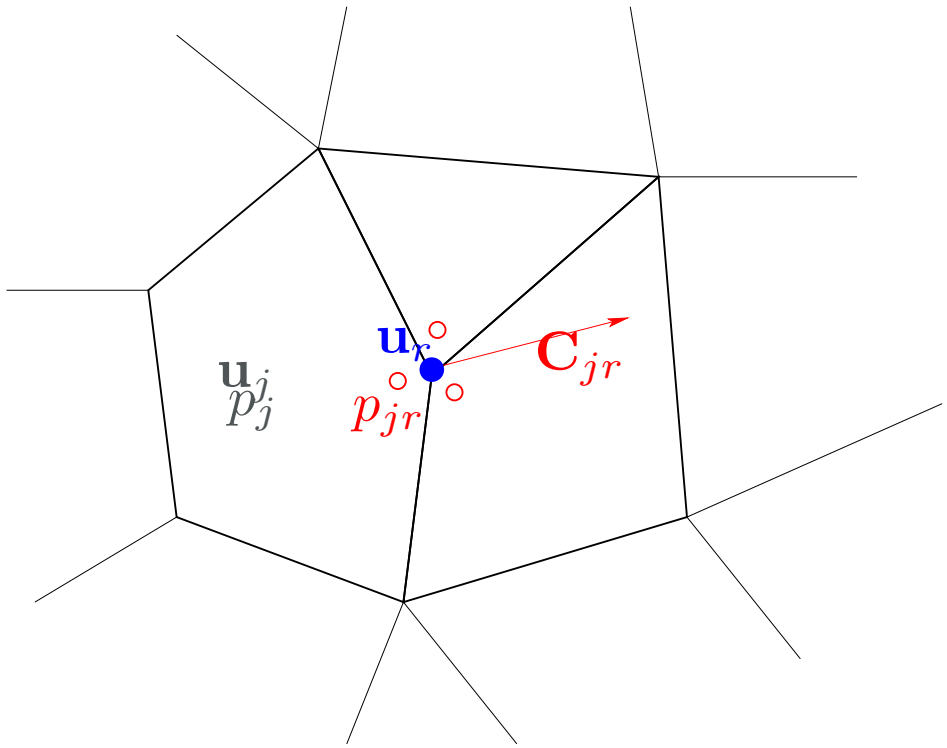
We have also

$$\sum_r (\mathbf{C}_{jr}, \mathbf{u}_r) p_r \approx \int_{\partial j} (\mathbf{u}, \mathbf{n}) p \, d\sigma = \int_{\partial j} (\mathbf{u}, p\mathbf{n}) \, d\sigma \left(= \int_j \nabla \cdot (p\mathbf{u}) \, dx \right).$$

This gives the structure of the Finite Volume Lagrangian scheme

$$\begin{cases} V_j'(t) = \sum_r (\mathbf{C}_{jr}, \mathbf{u}_r), \\ M_j \mathbf{u}_j'(t) = -\sum_r \mathbf{C}_{jr} p_{jr}, \\ M_j e_j'(t) = -\sum_r (\mathbf{C}_{jr}, \mathbf{u}_r) p_{jr}. \end{cases}$$

Construction of the fluxes



Riemann-invariant-like formula: $p_{jr} - p_j + \rho_j c_j \left(\mathbf{u}_r - \mathbf{u}_j, \mathbf{n}_{jr}\right) = 0.$

Sum of forces is locally zero: $\sum_j \mathbf{C}_{jr} p_{jr} = \sum_j |\mathbf{C}_{jr}| \, \mathbf{n}_{jr} p_{jr} = 0.$

There is a unique $\mathbf{u}_r$ solution magic formulas.

Solution

After elimination of the p_{jr}'s, the equation for $\mathbf{u}_r$ is

$$\left(\sum_j \rho_j c_j \left|\mathbf{C}_{jr}\right| \mathbf{n}_{jr} \otimes \mathbf{n}_{jr}\right) \mathbf{u}_r$$

$$= \left(\sum_j \mathbf{C}_{jr} p_j\right) + \left(\sum_j \rho_j c_j \left|\mathbf{C}_{jr}\right| \mathbf{n}_{jr} \otimes \mathbf{n}_{jr} \mathbf{u}_j\right).$$

Since the matrix is symetric and positive

$$A_r = \left(\sum_j \rho_j c_j \left|\mathbf{C}_{jr}\right| \mathbf{n}_{jr} \otimes \mathbf{n}_{jr}\right) = A_r^t > 0$$

there is is a unique solution $\mathbf{u}_r$. Then we compute the p_{jr}'s thanks to

$$p_{jr} - p_j + \rho_j c_j \left(\mathbf{u}_r - \mathbf{u}_j, \mathbf{n}_{jr}\right) = 0.$$

Scheme in 3D

- Cell quantities ($M_j, V_j, \mathbf{x}_j, \rho_j, e_j, \mathbf{u}_j$ and p_j) are given
- Compute $\mathbf{C}_{jr}$ on the current configuration
- We take $\Delta t \leq \min_j \left(\frac{\sum_r |\mathbf{C}_{jr}|}{3V_j} c_j \right)$.
- $A_r = \sum_j \rho_j c_j \, \mathbf{C}_{jr} \otimes \mathbf{n}_{jr} = A_r^t > 0.$
- $\mathbf{u}_r = A_r^{-1} \sum_j (\mathbf{C}_{jr} p_j + \rho_j c_j \, \mathbf{C}_{jr} \otimes \mathbf{n}_{jr} \, \mathbf{u}_j).$
- $p_{jr} = p_j + \rho_j c_j \left(\mathbf{u}_j - \mathbf{u}_r, \mathbf{n}_{jr} \right)$
- $\mathbf{u}_j^{n+1} = \mathbf{u}_j^n - \frac{\Delta t}{M_j} \sum_r \mathbf{C}_{jr} p_{jr}$
- $e_j^{n+1} = e_j^n - \frac{\Delta t}{M_j} \sum_r \left(\mathbf{C}_{jr}, \mathbf{u}_r \right) p_{jr}$
- $\mathbf{x}_r^{n+1} = \mathbf{x}_r^n + \Delta t \, \mathbf{u}_r$
- Update V_j, ρ_j and p_j

Conservativity

- Mass conservation: the scheme imposes M_j to be constant.
- Momentum conservation:

$$\begin{aligned}\sum_j M_j \mathbf{u}_j^{n+1} &= \sum_j M_j \mathbf{u}_j^n - \sum_j \Delta t \sum_r \mathbf{C}_{jr} p_r \\ &= \sum_j M_j \mathbf{u}_j^n - \Delta t \sum_r \sum_j \mathbf{C}_{jr} p_r \\ &= \sum_j M_j \mathbf{u}_j^n\end{aligned}$$

- Total energy conservation:

$$\begin{aligned}\sum_j M_j e_j^{n+1} &= \sum_j M_j e_j^n - \sum_j \Delta t \sum_r (\mathbf{C}_{jr}, \mathbf{u}_r)\, p_r \\ &= \sum_j M_j e_j^n - \Delta t \sum_r \left(\sum_j \mathbf{C}_{jr} p_r, \mathbf{u}_r \right) \\ &= \sum_j M_j e_j^n\end{aligned}$$

$\Longrightarrow$ the scheme is **locally** conservative.

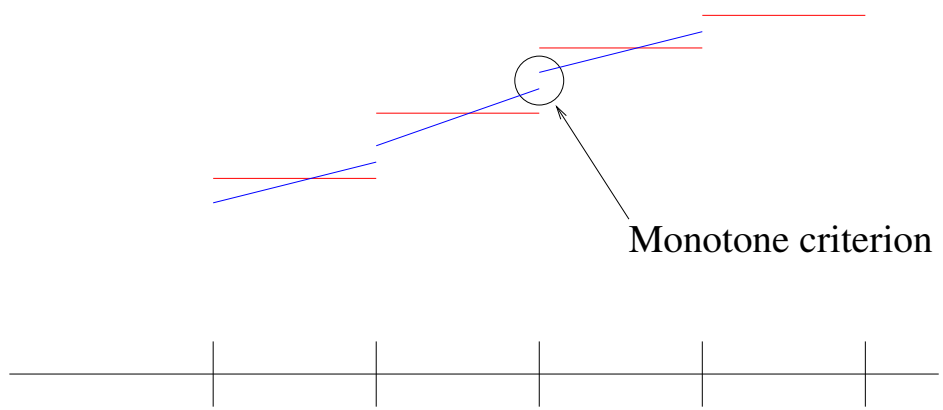
It is also stable and consistant.

Second order extension

- Goals: Improve the accuracy of the scheme and diminish scheme dissipation.
- Method: Muscl reconstruction + Van-Leer slope limiter:

$$\bar{\mathbf{u}}_j(\mathbf{x}) = \mathbf{u}_j + \Phi^{\mathrm{vl}} \nabla \mathbf{u}_j (\mathbf{x} - \mathbf{x}_j), \quad \bar{p}_j(\mathbf{x}) = p_j + \Phi^{\mathrm{vl}} \nabla p_j \cdot (\mathbf{x} - \mathbf{x}_j)$$

$\Phi^{\mathrm{vl}} \in [0, 1]$ is chosen so that $\bar{p}_j$ and $|\bar{\mathbf{u}}_j|$ respect some maximum principle *à la Ducowicz*.



The scheme is order 1 in time, so we need smaller time steps to avoid the anti-dissipative term of the equivalent equation

Second order extension

- Second Method. Muscl reconstruction + LW procedure + Van-Leer slope limiter (Dukovicz):
  - $\bar{\mathbf{u}}_j(\mathbf{x}) = \mathbf{u}_j + \Phi^{\mathrm{vl}}(1-\nu)\nabla\mathbf{u}_j(\mathbf{x}-\mathbf{x}_j)$
  - $\bar{p}_j(\mathbf{x}) = p_j + \Phi^{\mathrm{vl}}(1-\nu)\nabla p_j \cdot (\mathbf{x}-\mathbf{x}_j)$

Second order fluxes are still

- $\mathbf{u}_r = A_r^{-1} \sum_j (\mathbf{C}_{jr}\bar{p}_j(\mathbf{x}_r) + \rho_j c_j\, \mathbf{C}_{jr} \otimes \mathbf{n}_{jr}\, \bar{\mathbf{u}}_j(\mathbf{x}_r)).$
- $p_{jr} = \bar{p}_j(\mathbf{x}_r) + \rho_j c_j\, (\bar{\mathbf{u}}_j(\mathbf{x}_r) - \mathbf{u}_r, \mathbf{n}_{jr})$

The scheme is formally order 2 in time and space for simple 1D problems

Numerical results

We compute the results of some basic shock tube problems in various geometries

Sod shock tube

$$\gamma = 1.4, u = 0 \qquad \begin{cases} p(x) = 1 & \rho(x) = 1 & \text{if } x < 0.5, \\ p(x) = 0.1 & \rho(x) = 0.125 & \text{else.} \end{cases}$$

Shestakov shock tube

$$\gamma = \tfrac{5}{3} \quad u = 0 \quad p = \tfrac{2}{3} \times 10^{14} \quad x_{min}(0) = 0 \quad x_{max}(0) = 0.75$$

$$u(x_{max}(t)) = 2.5 \times 10^7$$

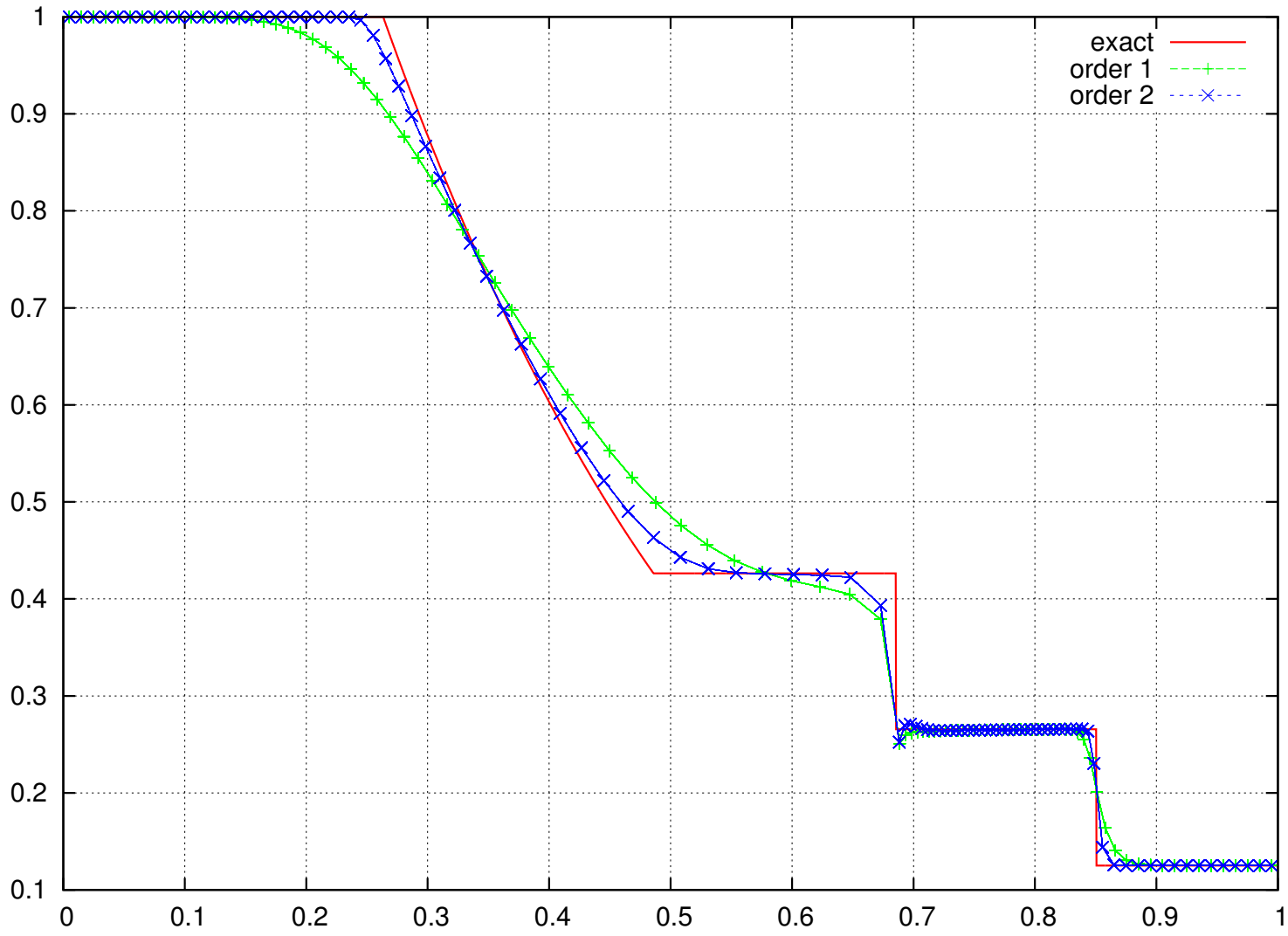
$$u(x_{min}(t)) = \begin{cases} 10^8 \left((1 - 10^8 t)^{-0.25} - 1 \right), & \text{if } 0 \leq t < 0.9 \times 10^{-8}, \\ 10^8 (10^{0.25} - 1) \approx 7.8 \times 10^7 & \text{if } 0.9 \times 10^{-8} \leq t < 10^{-8}, \\ 0 & \text{else.} \end{cases}$$

Sod shock tube in 2D and 3D

$$\gamma = 1.4, u = 0 \qquad \begin{cases} p(r) = 1 & \rho(r) = 1 & \text{if } x > 0.5, \\ p(r) = 0.1 & \rho(r) = 0.125 & \text{else.} \end{cases}$$

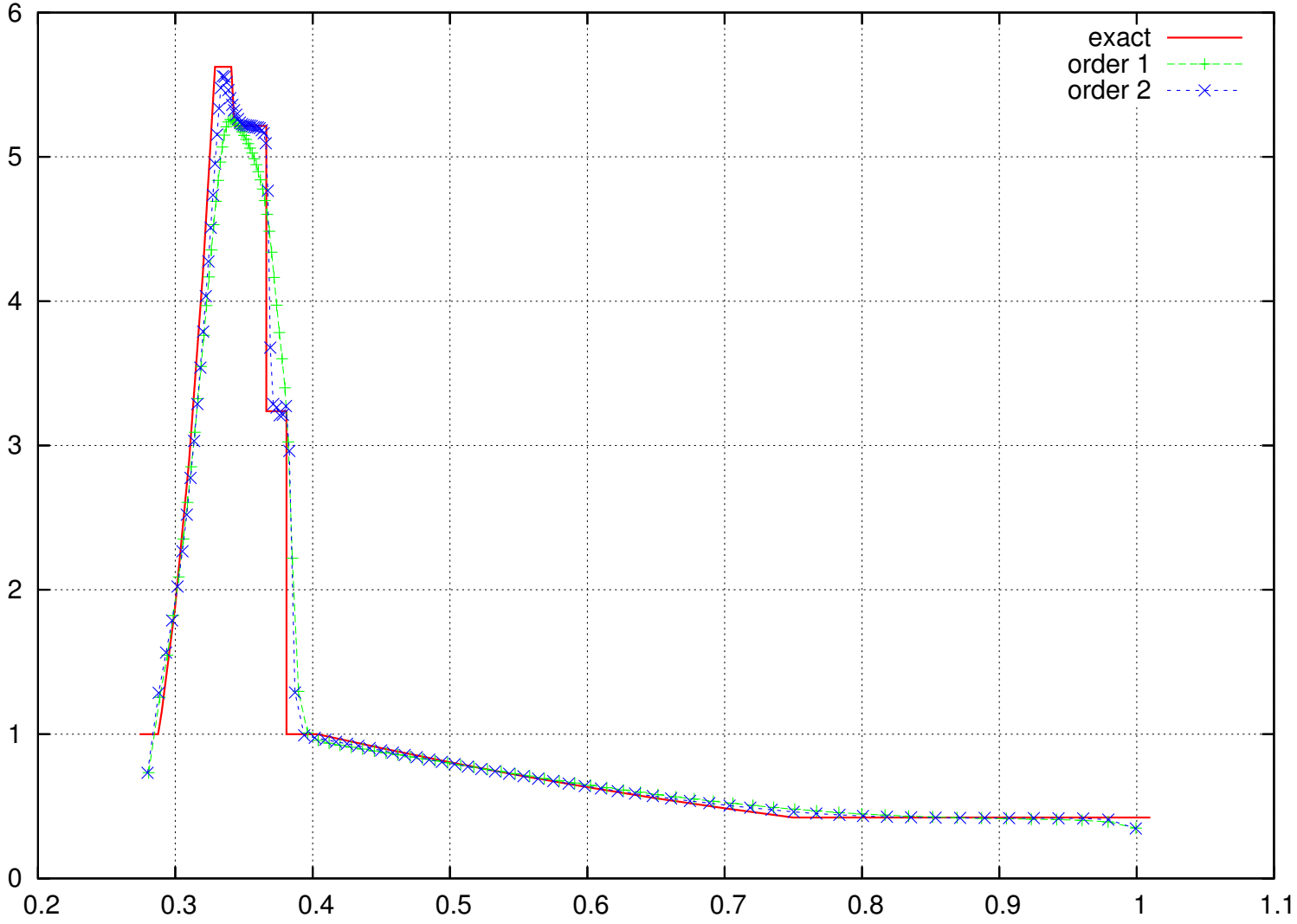
Sod shock tube - 1D

Density 100 cells, $t = 0.2s$



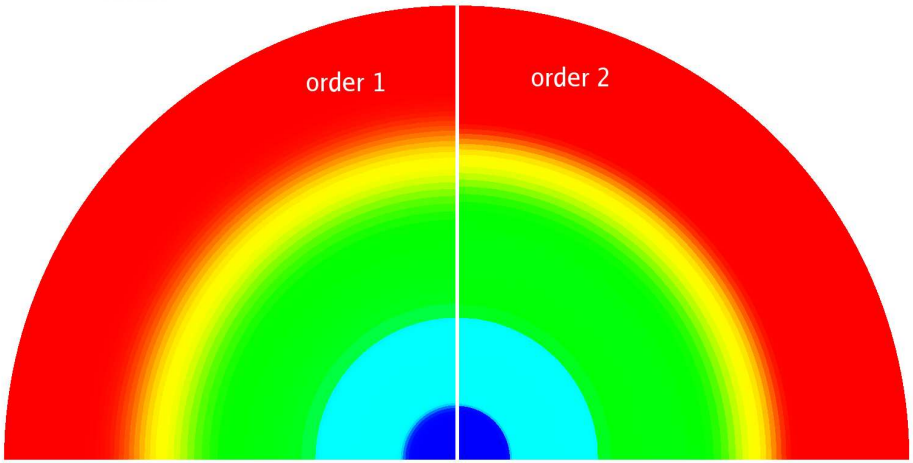
Shestakov shock tube - 1D

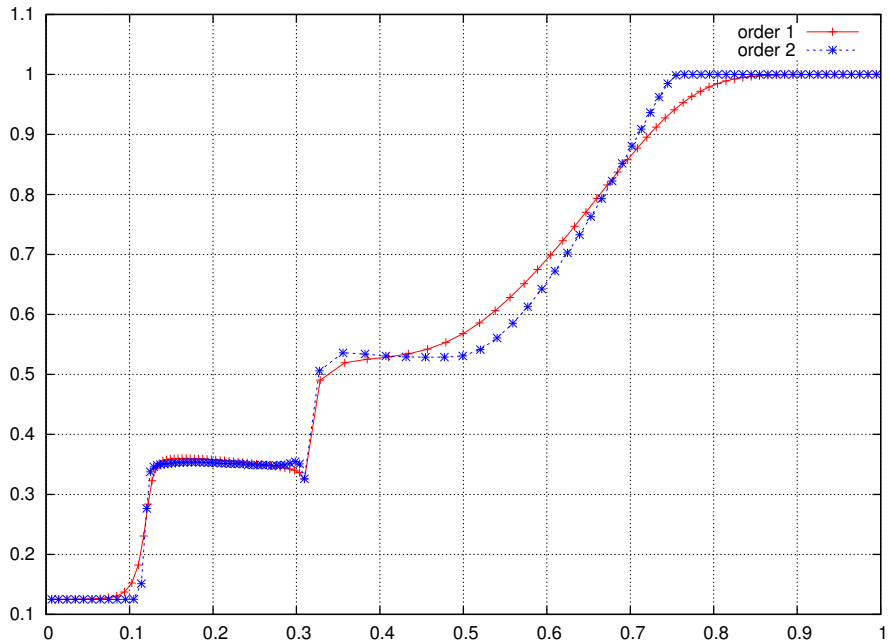
Density 100 cells, $t = 10.4 \times 10^{-9} s$



Sod shock tube - 2D

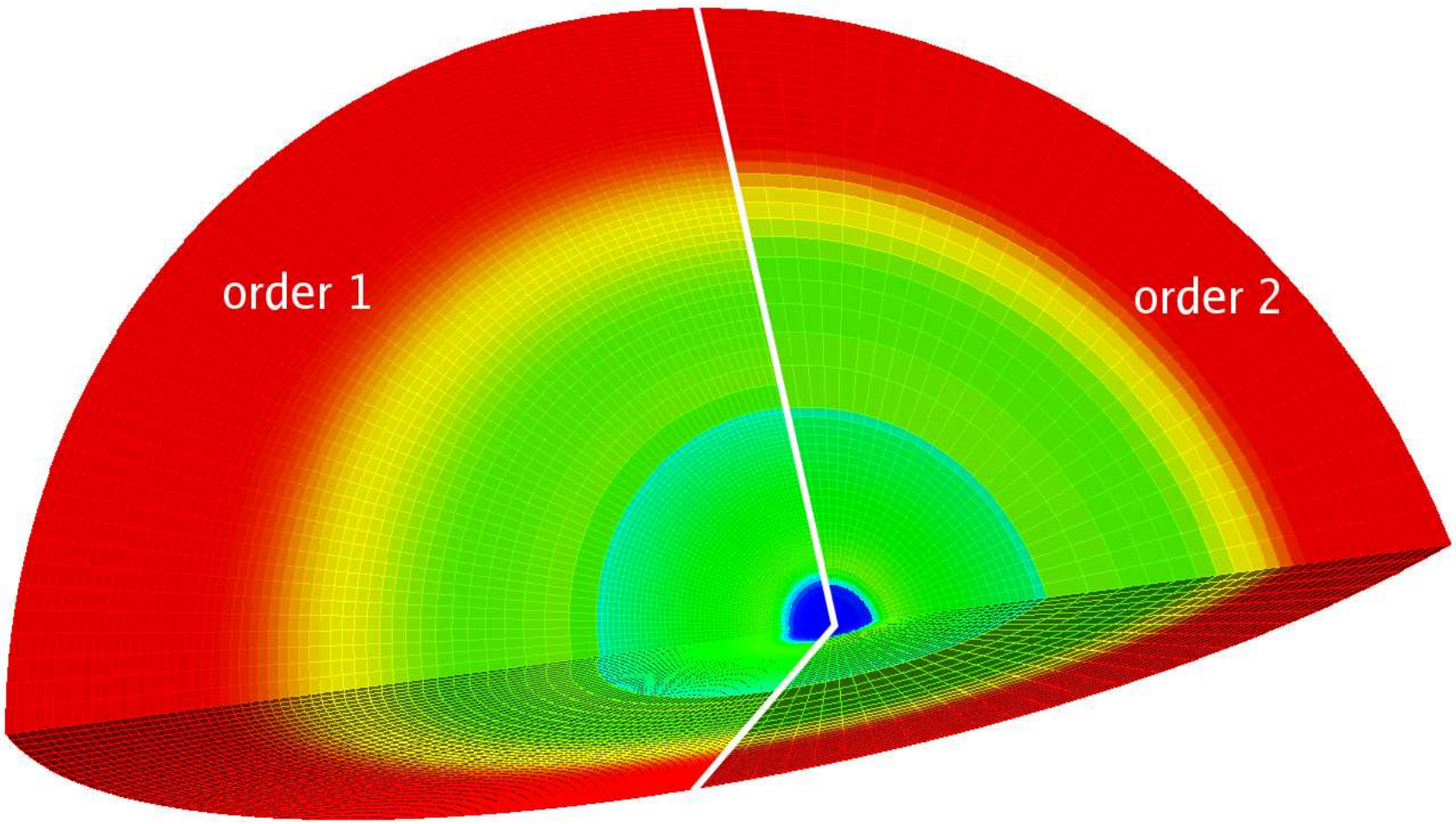
Density 100×50 cells, $t = 0.2s$





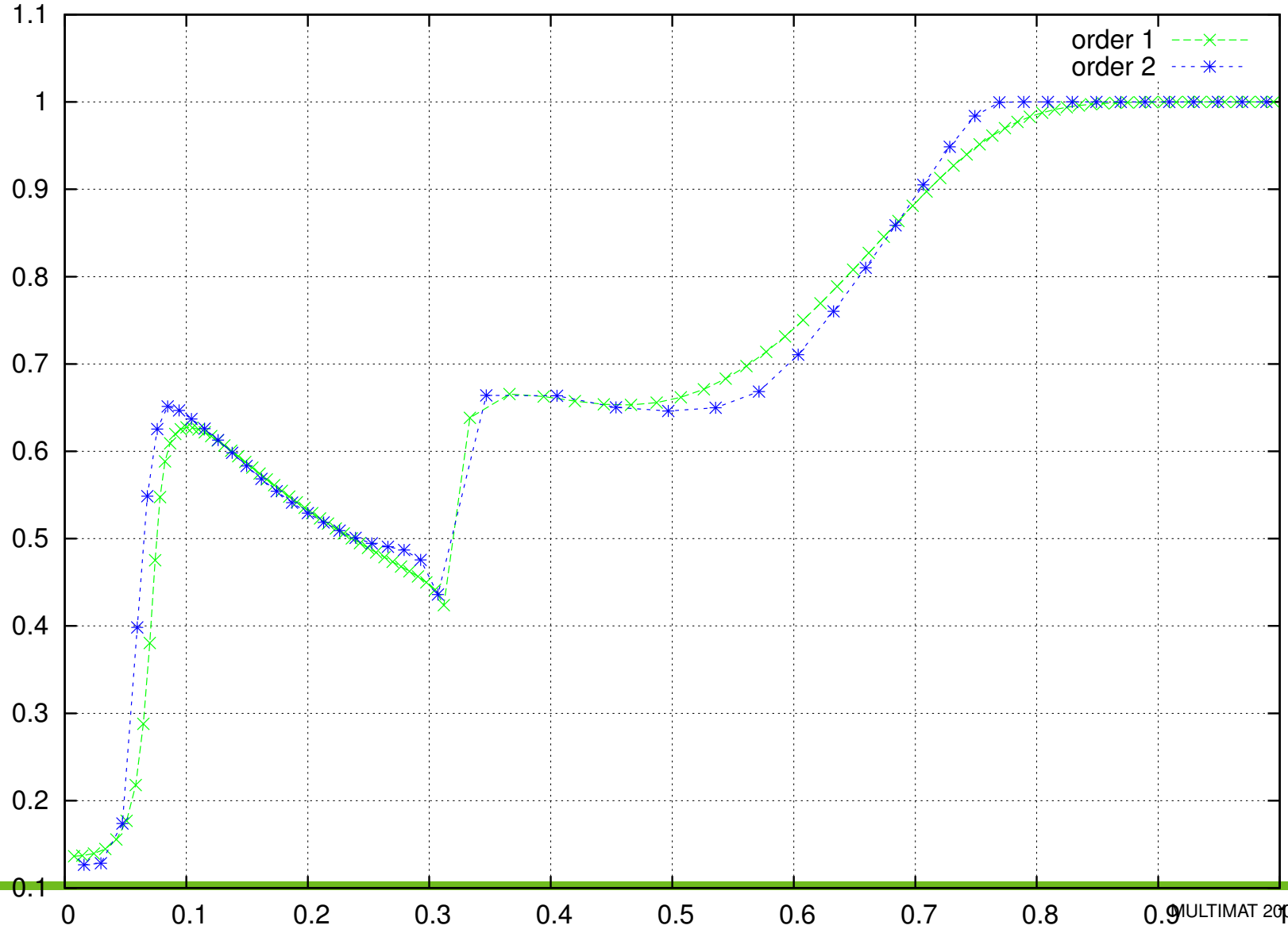
Sod shock tube - 3D

Density. Order one $187\,500$. Order two $21\,600$ cells. $t = 0.2s$. Parallel computation



Sod shock tube - 3D

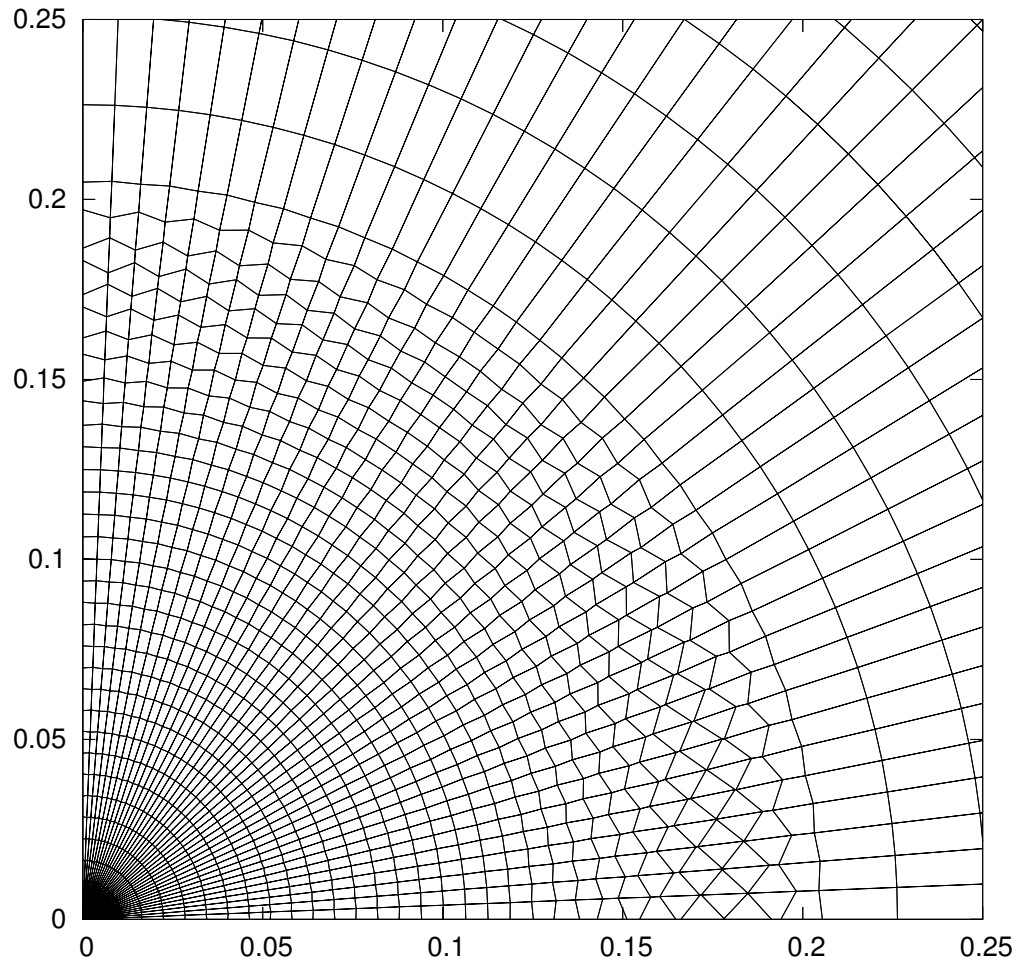
Density profile



Hourglass with Noh

This has already been identified in 2D.

Noh problem in 2D: $\gamma = \frac{5}{3}$, $\rho = 1$, $p = 0 (\approx 10^{-6})$, $\mathbf{u} = -\mathbf{e}_r$.
Using a polar mesh, one obtains

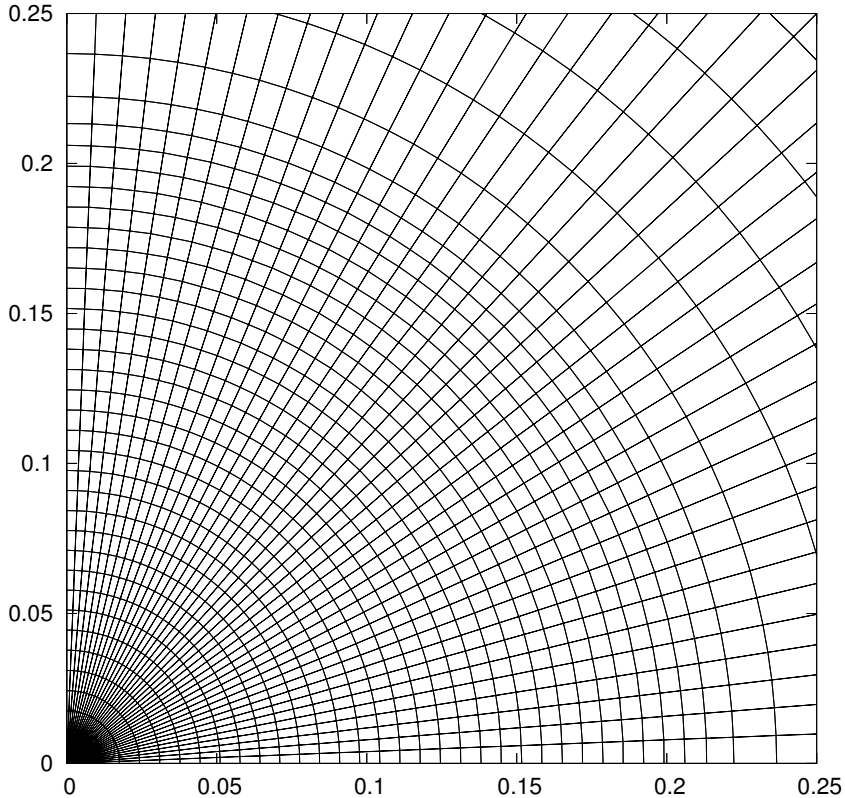


A fix-up

- $c_j \ll 1 \implies A_r$ nearly singular.
- Noh: $c_j \approx 10^{-3} \implies A_r = \sum_j \rho_j c_j \, \mathbf{C}_{jr} \otimes \mathbf{n}_{jr} \approx 0$.

$$\mathbf{u}_r = A_r^{-1} \sum_j (\mathbf{C}_{jr} \bar{p}_j(\mathbf{x}_r) + \rho_j c_j \, \mathbf{C}_{jr} \otimes \mathbf{n}_{jr} \, \bar{\mathbf{u}}_j(\mathbf{x}_r)).$$

- Idea: increase c when it is too small for the fluxes calculus (more dissipative scheme).



Time step optimization

In order to increase the time step, we can replace

$$p_{jr} - p_j + \rho_j c_j \left(\mathbf{u}_r - \mathbf{u}_j, \mathbf{n}_{jr}\right) = 0$$

by

$$p_{jr} - p_j + \lambda_j \rho_j c_j \left(\mathbf{u}_r - \mathbf{u}_j, \mathbf{n}_{jr}\right) = 0.$$

It can be shown that the optimal λ_j is

$$\lambda_j = \sum_r \frac{|\mathbf{C}_{jr}|}{\Lambda} \in [1, \sqrt{3}],$$

where Λ is the maximum eigenvalue of

$$\sum_r \frac{\mathbf{C}_{jr} \otimes \mathbf{C}_{jr}}{|\mathbf{C}_{jr}|}.$$

Finally $\Delta t \leq \min_j \left(\lambda_j \frac{\sum_r |\mathbf{C}_{jr}|}{3V_j} c_j \right)$

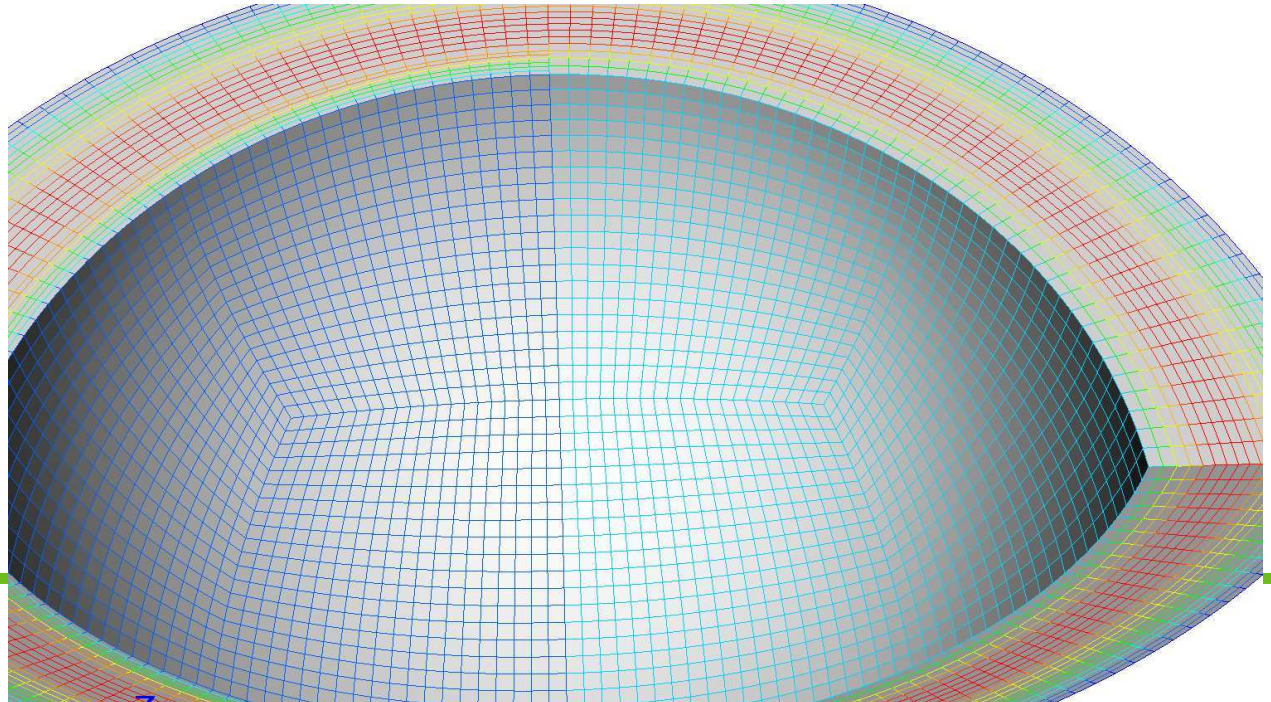
Reduce connectivity points in 3D

- The low-Mach test case is similar to the 2D one published in the thesis of Raphael Loubere

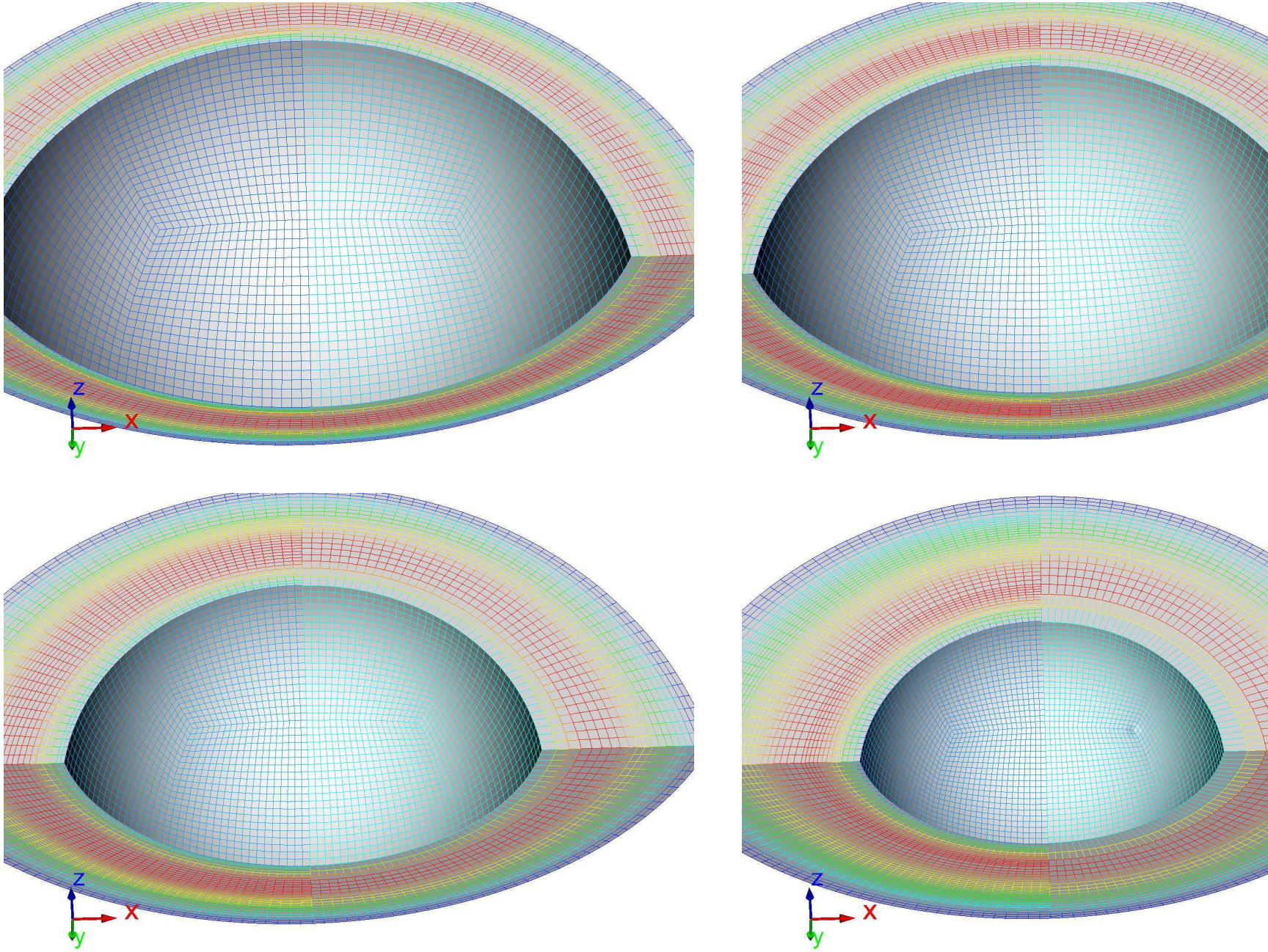
$$|u| << c.$$

  It consits of a 3D shell implosion, computed with a perfect gas law, and with a prescribed pressure on the internal and external faces of the shell. Since the computation is done with $\frac{1}{8}$ of the shell, sliding walls are used on other boundaries.

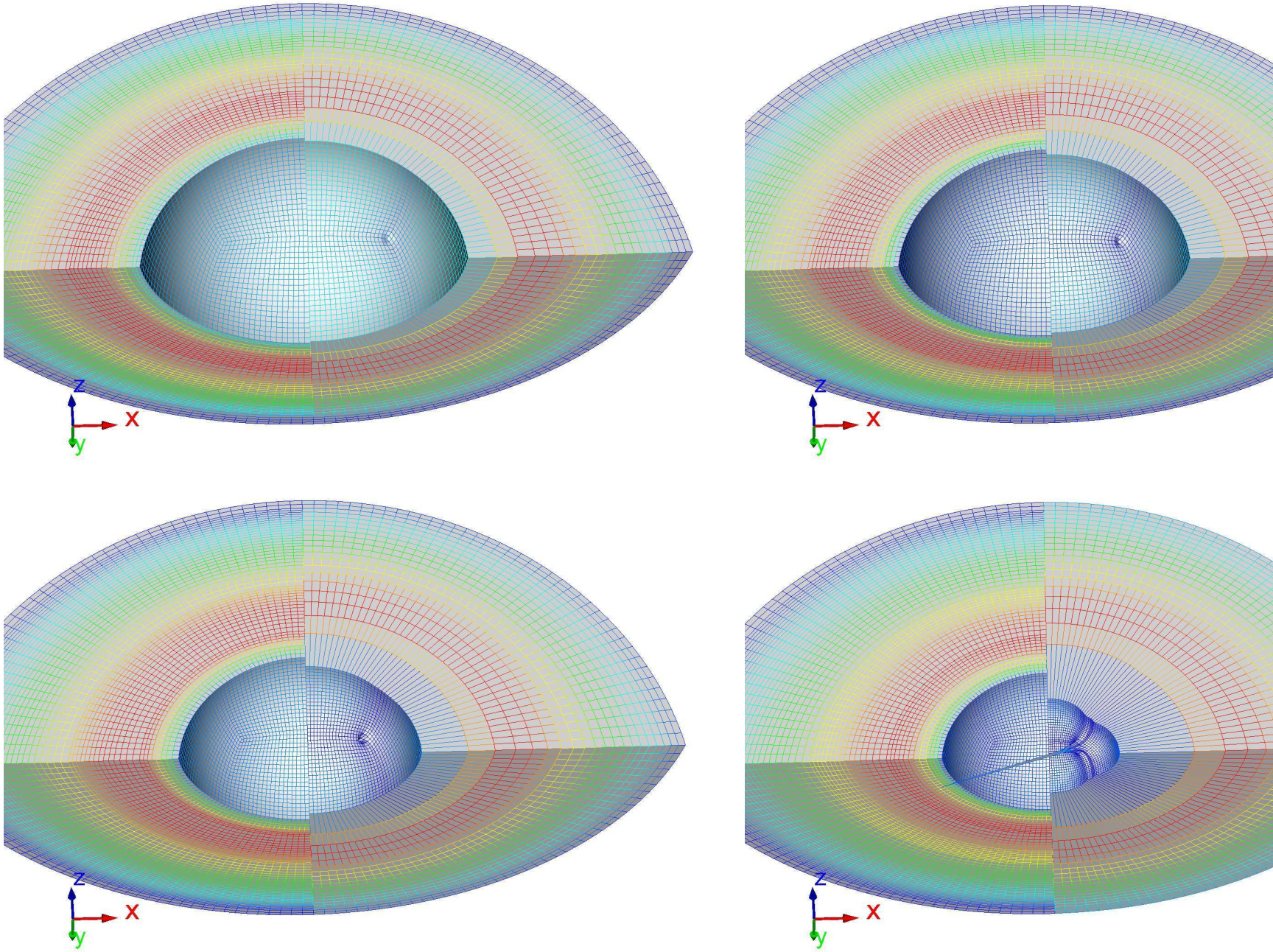
- The reduced connectivity point is important.



Reduce connectivity points in 3D

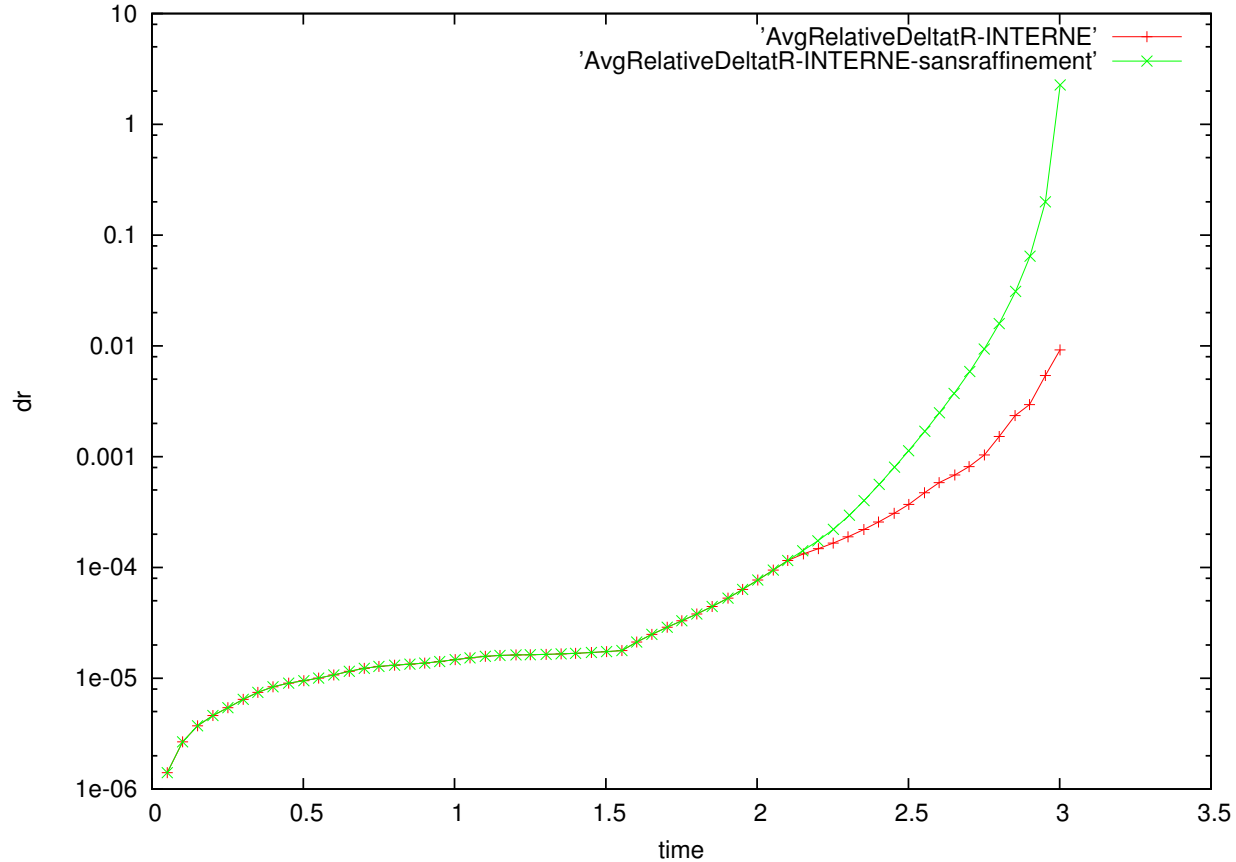


Reduce connectivity points in 3D



Reduce connectivity points in 3D

$t \mapsto d\overline{r}(t)$ is the sphericity error compute on the internal boundary.



AMR (and ALE) techniques are easy to use for cell-centered schemes.
For this problem it enhances the final quality and exerces some control on the reduce connectivity point.

Conclusions et perspectives

- The whole scheme relies on the definition of $\mathbf{C}_{jr}$.
  - This is straightforward only for tetrahedrons.
  - No unique definition of $\mathbf{C}_{jr}$ for other meshes in 3D.
  - So we need to compare all possible definition of theses quantities.
- Boundary conditions and second order Muscl reconstruction completed.
- We begin the evaluation of the coupling with ALE and remeshing.
- The reduce connectivity point (in 3D) behaves well in conjunction with AMR.
- We shall start coupling with other physics (diffusion) to test this scheme for ICF simulations.
- ...