
A pure eulerian scheme for multi-material fluid flows

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GOALS:

- Second order scheme in 2D-3D
- Cartesian structured meshes, plane or axi-symmetric
- Robust scheme
- Fully conservative method
- Multi-material computations with real fluid EOS
- Interface capturing in 2D-3D
- Enable materials to slip by each others within a cell
**SINGLE-MATERIAL : FINITE VOLUME FVCF SCHEME**

- **VFFC : Finite Volume with Characteristic Flux** (Ghidaglia et al. [1])

The vector of quantities $V$ per volume unit is cell centered.

$$V = \begin{pmatrix} \rho \\ \rho u \\ \rho E \end{pmatrix}$$

$$F \cdot n = \sum_{i=1}^{d} F_{i} n_{i} = \begin{pmatrix} \rho (u \cdot n) \\ \rho u (u \cdot n) + pn \\ \rho E (u \cdot n) + p(u \cdot n) \end{pmatrix} = V(u \cdot n) + p N$$

with $N = (0, n, u \cdot n)'$,

- $\rho$ the density, $u$ the velocity in $\mathbb{R}^d$,
- $e$ the specific internal energy,
- $E = e + 1/2 \ u^2$ the specific total energy,
- $p = P(\rho, e)$ the pressure for a given EOS.

The finite volume scheme reads:

$$\frac{\partial}{\partial t} \int_{\Omega} V \ d\Omega + \int_{\Gamma} F \cdot \bar{n} \ d\Gamma = 0$$

$$\Rightarrow \ Vol \ \frac{V^{n+1} - V^n}{dt} + \sum_{i=1}^{2d} A_{i} \phi_{i} = 0$$
The FVCF flux through the face between $C_L$ and $C_R$ is obtained by upwinding the normal fluxes in the diagonal space:

$$\phi(V_L, V_R, n_{LR}) = \frac{F_L + F_R}{2} \cdot n_{LR} + \text{sign}\left( J(V_\mu, n_{LR}) \right) \frac{F_L - F_R}{2} \cdot n_{LR}$$

with $\text{diag}(\lambda_i) = L \ J(V_\mu, n) R$ and $\text{sign}\left( J(V_\mu, n) \right) = R \ \text{diag} \ (\text{sign}(\lambda_i)) L$
• **NIP : Natural Interface Positioning**
  • 1D interface capturing, two main difficulties :

  • How to manage the time step CFL condition with very small volumes and how goes the interface through a cell face ?

  • How to define the flux $\phi^{\text{int}}$ at $x_{\text{int}}$ between two different materials, with different EOS ?

![Diagram showing interface capturing in a multi-material system](image)
• How to manage the time step CFL condition with very small volumes and how goes the interface through a cell boundary?
• How to define the flux at $x_{\text{int}}$ between two different materials, with different EOS?

• In dimension one, we integrate the eulerian system on a moving volume $\Omega(t)$:

$$\int_{\Omega(t)} \frac{\partial V}{\partial t} + \frac{\partial F}{\partial x} \, d\Omega = 0$$

It comes:

$$\int_{\Omega(t)} \frac{\partial V}{\partial t} \, d\Omega = \frac{\partial}{\partial t} \int_{\Omega(t)} V \, d\Omega - \int_{\Gamma(t)} V(\vec{u}_\Gamma \cdot \vec{n}_\Gamma) \, d\Gamma$$

$$\int_{\Omega(t)} \frac{\partial F}{\partial x} \, d\Omega = \int_{\Gamma(t)} F \cdot \vec{n}_\Gamma \, d\Gamma$$

The eulerian normal flux reads:

$$F \cdot \vec{n}_\Gamma = V(\vec{u}_\Gamma \cdot \vec{n}_\Gamma) + p_\Gamma N_\Gamma$$

$$\frac{\partial}{\partial t} \int_{\Omega(t)} V \, d\Omega + \int_{\Gamma(u_\Gamma = 0)} F \cdot \vec{n}_\Gamma \, d\Gamma + \int_{\Gamma(u_\Gamma \neq 0)} p_\Gamma \cdot N_\Gamma \, d\Gamma = 0$$

with $N_\Gamma = (0, \vec{n}_\Gamma, \vec{u}_\Gamma \cdot \vec{n}_\Gamma)^t$
multi-material : INTERFACE CAPTURING NIP

- We write the conservation laws on partial volumes left (L) and right (R)
  - \( \phi^L \) and \( \phi^R \) are calculated by the FVCF scheme

\[
\begin{align*}
\frac{Vol_{L}^{n+1} V_{L}^{n+1} - Vol_{L}^{n} V_{L}^{n}}{dt} + A \left( \phi^L + p_{\text{int}} N_{\text{int}L} \right) &= 0 \quad \text{with} \quad N_{\text{int}L} = (0, \bar{n}_{L}, \bar{u}_{\text{int}} \cdot \bar{n}_{L}) \\
\frac{Vol_{R}^{n+1} V_{R}^{n+1} - Vol_{R}^{n} V_{R}^{n}}{dt} + A \left( \phi^R + p_{\text{int}} N_{\text{int}R} \right) &= 0 \quad \text{with} \quad N_{\text{int}R} = (0, \bar{n}_{R}, \bar{u}_{\text{int}} \cdot \bar{n}_{R})
\end{align*}
\]

\[
\begin{align*}
m_{L}^{n+1} &= m_{L}^{n} - dt A \phi^L \\
u_{L}^{n+1} &= \frac{m_{L}^{n}}{m_{L}^{n+1}} u_{L}^{n} - \frac{dt A}{m_{L}^{n+1}} (\phi^L + p_{\text{int}}) \\
v_{L}^{n+1} &= \frac{m_{L}^{n}}{m_{L}^{n+1}} v_{L}^{n} - \frac{dt A}{m_{L}^{n+1}} \phi^L \\
E_{L}^{n+1} &= \frac{m_{L}^{n}}{m_{L}^{n+1}} E_{L}^{n} - \frac{dt A}{m_{L}^{n+1}} (\phi^L + p_{\text{int}} u_{\text{int}}) \\
m_{R}^{n+1} &= m_{R}^{n} - dt A \phi^R \\
u_{R}^{n+1} &= \frac{m_{R}^{n}}{m_{R}^{n+1}} u_{R}^{n} - \frac{dt A}{m_{R}^{n+1}} (\phi^R - p_{\text{int}}) \\
v_{R}^{n+1} &= \frac{m_{R}^{n}}{m_{R}^{n+1}} v_{R}^{n} - \frac{dt A}{m_{R}^{n+1}} \phi^R \\
E_{R}^{n+1} &= \frac{m_{R}^{n}}{m_{R}^{n+1}} E_{R}^{n} - \frac{dt A}{m_{R}^{n+1}} (\phi^R - p_{\text{int}} u_{\text{int}})
\end{align*}
\]

\[
\begin{align*}
\phi^L &\xrightarrow{u_{\text{int}}} \quad P_{\text{int}} N_{\text{int}R} \\
L &\xleftarrow{n_{\text{int}R}} \quad \bar{n}_{R} \\
\phi^L &\xrightarrow{n_{\text{int}L}} \quad P_{\text{int}} N_{\text{int}L} \\
R &\xleftarrow{n_{\text{int}L}} \quad \bar{n}_{L}
\end{align*}
\]
The 1D approach will be extended in 2D/3D by directional splitting

- Single-material:
  - Directional splitting does not change the scheme if fluxes on each directions are computed with values at the same time $t^n$

$$Vol \frac{V^{n+1} - V^n}{dt} + A_x (\phi_x^{\text{Right},n} + \phi_x^{\text{Left},n}) + A_y (\phi_y^{\text{Up},n} + \phi_y^{\text{Down},n}) + A_z (\phi_z^{\text{Front},n} + \phi_z^{\text{Rear},n}) = 0$$
2D EXTENSION

- Extension to the multi-material case in 2D/3D
  - Phase $x$

![Diagram showing a 2D extension with a condensate region labeled.](image-url)
2D EXTENSION

- Multi-material:
  - Directions are splitted, using the same 1D algorithm in each direction
  - At the beginning of the computation, you know the order of the materials within a mixed cell in each direction. This order is kept by the method.
  - Condensation of mixed cells if one have to deal with neighbouring mixed cells.

![Diagram showing 1D conservative projection, condensate, remap, and interface positioning]
2D EXTENSION

- Interface pressure and velocity in 1D:
  - \textit{FVCF} is written in lagrangian coordinates
  - Calculation of two local Riemann invariants with equations:
    \[
    \frac{\partial \pi^+}{\partial t} + c \frac{\partial \pi^+}{\partial x} = 0, \quad \text{with} \quad \pi^+ = p + \rho c u
    \]
    \[
    \frac{\partial \pi^-}{\partial t} - c \frac{\partial \pi^-}{\partial x} = 0, \quad \text{with} \quad \pi^- = p - \rho c u
    \]
  - Two conditions to obtain pressure and velocity:
    \[
    \pi_{\text{int}}^+ = \pi_{l}^+ \quad \iff \quad p_{\text{int}} + \rho_l c_l u_{\text{int}} = p_l + \rho_l c_l u_l
    \]
    \[
    \pi_{\text{int}}^- = \pi_{r}^- \quad \iff \quad p_{\text{int}} - \rho_r c_r u_{\text{int}} = p_r - \rho_r c_r u_r
    \]
  - The solution is:
    \[
    \begin{align*}
    p_{\text{int}} &= \frac{\rho_r \tilde{c}_r p_l + \rho_l \tilde{c}_l p_r}{\rho_l \tilde{c}_l + \rho_r \tilde{c}_r} + \rho_l \tilde{c}_l \rho_r \tilde{c}_r \frac{u_l - u_r}{\rho_l \tilde{c}_l + \rho_r \tilde{c}_r} \\
    u_{\text{int}} &= \frac{\rho_l \tilde{c}_l u_l + \rho_r \tilde{c}_r u_r}{\rho_l \tilde{c}_l + \rho_r \tilde{c}_r} + \frac{p_l - p_r}{\rho_l \tilde{c}_l + \rho_r \tilde{c}_r}
    \end{align*}
    \]
    with \( \tilde{c}_i = \min\left( c_i, \frac{dx_i}{dt} \right) \) \( \Rightarrow \) \( T_{ds_i} = e_i^{n+1} - e_i^n + p_i^n \left( \frac{1}{\rho_i^{n+1}} - \frac{1}{\rho_i^n} \right) \geq 0 \)
2D EXTENSION

- 2D interface pressure gradient and velocity are obtained by writing 1D equations in direction of the interface normal vector, in order to impose perfect sliding:

\[
\nabla p \cdot n = p_{\text{int}} - \frac{\rho_r \tilde{c}_r p_l + \rho_l \tilde{c}_l p_r}{\rho_l \tilde{c}_l + \rho_r \tilde{c}_r} = \rho_l \tilde{c}_l \rho_r \tilde{c}_r \frac{u_l - u_r}{\rho_l \tilde{c}_l + \rho_r \tilde{c}_r} \cdot n
\]

\[
u_{\text{int}} \cdot n = \left( \frac{\rho_l \tilde{c}_l u_l + \rho_r \tilde{c}_r u_r}{\rho_l \tilde{c}_l + \rho_r \tilde{c}_r} \cdot n + \frac{p_l - p_r}{\rho_l \tilde{c}_l + \rho_r \tilde{c}_r} \right)
\]

- In phase \( x \) of the directional splitting, we only consider the \( x \) component of pressure gradient and velocity:

\[
\begin{align*}
p_{\text{int}}^x &= \frac{\rho_r \tilde{c}_r p_l + \rho_l \tilde{c}_l p_r}{\rho_l \tilde{c}_l + \rho_r \tilde{c}_r} + \left( \frac{\rho_l \tilde{c}_l \rho_r \tilde{c}_r}{\rho_l \tilde{c}_l + \rho_r \tilde{c}_r} \frac{u_l - u_r}{\rho_l \tilde{c}_l + \rho_r \tilde{c}_r} \cdot n \right) n^x \\
u_{\text{int}}^x &= \frac{\rho_l \tilde{c}_l u_l^x + \rho_r \tilde{c}_r u_r^x}{\rho_l \tilde{c}_l + \rho_r \tilde{c}_r} + \frac{p_l - p_r}{\rho_l \tilde{c}_l + \rho_r \tilde{c}_r} n^x
\end{align*}
\]

- Unfortunately, we did not find 2D/3D expressions of \( p_{\text{int}} \) and \( u_{\text{int}} \) that set positive entropy dissipation for each phase of the directional splitting.
In small layers in the condensate, the CFL condition is not fulfilled!

- What is CFL condition?
  - In the case of linear advection, using a basic upwind scheme:

\[
\frac{u_i^{n+1} - u_i^n}{dt} + a \frac{u_i^n - u_{i-1}^n}{dx} = 0 \quad \text{with } a > 0
\]

\[\iff u_i^{n+1} = u_i^n \left(1 - \frac{a}{dt} \right) + \frac{a}{dt} u_{i-1}^n \quad \text{if } 0 < \frac{a}{dt} < 1, \text{ monotonicity}\]

\[\iff \frac{u_i^{n+1} - u_i^n}{u_i^n} = -\frac{a}{dt} \left(\frac{u_i^n - u_{i-1}^n}{u_i^n}\right) \quad \text{if } 0 < \frac{a}{dt} < 1, \text{ small variations of } u\]

- What is small variations and for which quantities?
2D EXTENSION

What is small variations and for which quantities?

- We choose to control variations of pressure $p$.
- Variation of $p$ is a function of variations of density and internal energy, given by:

\[
\rho \frac{dp}{d\rho} + (\frac{\partial p}{\partial \rho})_s \cdot \frac{\partial \rho}{\partial e} = \rho \frac{\partial e}{\partial \rho}
\]

Let us be $p = P(\rho, e)$ for a given EOS,

with $\rho$ the density, $e$ the specific internal energy,

\[
c^2 = \left( \frac{\partial p}{\partial \rho} \right)_s = \frac{p\Gamma}{\rho} + \left( \frac{\partial p}{\partial \rho} \right)_e
\]

sound velocity (square),

\[
\Gamma = \frac{1}{\rho} \left( \frac{\partial p}{\partial e} \right)_\rho
\]

Gruneisen coefficient,

\[
\Rightarrow dp = \rho \Gamma de + \left( c^2 - \frac{p\Gamma}{\rho} \right) d\rho
\]
2D EXTENSION

- Variation of $p$ in layer $i$ is a function of variations of density and internal energy of the form:

$$
\left| p_{i}^{n+1} - p_{i}^{n} \right| \approx \frac{\rho_{i}^{n} \Gamma_{i}^{n}}{\rho_{i}^{n}} \left( e_{i}^{n+1} - e_{i}^{n} \right) + \frac{\left( c_{i}^{n} \right)^{2} - \Gamma_{i}^{n}}{\rho_{i}^{n}} \left( \rho_{i}^{n+1} - \rho_{i}^{n} \right) \leq 2\varepsilon_{p}
$$

$$
\left| e_{i}^{n+1} - e_{i}^{n} \right| \leq \frac{\varepsilon_{p}}{\rho_{i}^{n} \Gamma_{i}^{n}} \quad \text{and} \quad \left| \rho_{i}^{n+1} - \rho_{i}^{n} \right| \leq \frac{\varepsilon_{p}}{\frac{\left( c_{i}^{n} \right)^{2} - \Gamma_{i}^{n}}{\rho_{i}^{n} \rho_{i}^{n}}}
$$

- Some analyses of the scheme give:

$$
\frac{\Delta(u)_{i}}{\lambda_{i}^{n}} \leq \frac{\varepsilon_{p} p_{i}^{n}}{\rho_{i}^{n} \left( c_{i}^{n} \right)^{2}}
$$

$$
\frac{\Delta(p)_{i}}{\rho_{i}^{n} \left( \lambda_{i}^{n} \right)^{2}} \leq \frac{2\varepsilon_{p} p_{i}^{n}}{\Gamma_{i}^{n} \rho_{i}^{n} \left( \lambda_{i}^{n} \right)^{2}}
$$

with $\lambda_{i}^{n} = \frac{dx_{i}^{n}}{dt}$, $\Delta(x)_{i} = x_{i+1/2} - x_{i-1/2}$ ($i + 1 / 2$ denotes quantities at interfaces)
2D EXTENSION

• At the end of each 1D calculation of the directional splitting, interfaces normal vectors are updated this way:

Interface positioning with respect to volume fraction (VOF) following D. L. Youngs [5]

\[ \vec{n} = F(f_1^i, f_2^i) \]

\[ \vec{n}_i = - \frac{\nabla f_i}{\| \nabla f_i \|} \]

• Notice that the explicit position of the interface is not needed in this method. Only the interface normal vector and materials ordering are needed.
1D RESULT

- Blastwave test case of Woodward & Collela [2] using 400 cells
- Comparison by Liska&Wendroff [3] of density profile at time $t=0.038$

$$\begin{align*}
\begin{cases}
  p = 1000 \\
  \rho = 1 \\
  u = 0
\end{cases} & \quad \begin{cases}
  p = 0.01 \\
  \rho = 1 \\
  u = 0
\end{cases} & \quad \begin{cases}
  p = 100 \\
  \rho = 1 \\
  u = 0
\end{cases}
\end{align*}$$
2D RESULTS

- Perfect shear test
  - Velocities at t=0 s

\[ U_x = 0.4, \quad U_y = 1. \]

\[ U_x = -0.4, \quad U_y = -1. \]
2D RESULTS

- Perfect shear test
  - Density at $t=0.25$
2D RESULTS

- Perfect shear test
  - Pressure at $t=0.25$
2D RESULTS

- Perfect shear test
- Velocities at t=0.25
2D RESULTS

- Sedov test case
  - Perfect gas EOS with specific heat ratio 1.66
  - Mesh is 100x100 cells

\[
\begin{align*}
P &= 1.5 \\
\rho &= 1. \\
U_x &= U_y = 0.
\end{align*}
\]

\[
\begin{align*}
P &= 5 \times 10^{19} \\
\rho &= 1. \\
U_x &= U_y = 0.
\end{align*}
\]
2D RESULTS

- Sedov test case
  - Pressure at $t = 2.5 \times 10^{-9}$ s
2D RESULTS

- Sedov test case
  - Kinetic energy $t = 2.5 \times 10^{-9}$ s
2D RESULTS

• Fall of a volume of water on the ground in air
  • EOS for air is assumed to be perfect gas:
    \[ p = (\gamma_{\text{air}} - 1) \rho e \quad \text{with} \quad \gamma_{\text{air}} = 1.4 \]
    \[ c_{\text{air}}^2 = \frac{\gamma_{\text{air}}}{\rho} \frac{p}{\gamma_{\text{air}} - 1} \]
    \[ \Gamma_{\text{air}} = \gamma_{\text{air}} - 1 \]

• EOS for water is assumed to be stiffened gas:
  \[ p = (\gamma_{\text{water}} - 1) \rho e - \pi \quad \text{with} \quad \gamma_{\text{water}} = 7, \pi = 21 \cdot 10^8 \text{ Pa} \]
  \[ c_{\text{water}}^2 = \frac{\gamma_{\text{water}} p + \pi}{\rho} \quad \text{and} \quad \Gamma_{\text{water}} = \gamma_{\text{water}} - 1 \]
2D RESULTS

• Fall of a volume of water on the ground in air
  • Initial state of air is $v=0$, density = $1 \text{ kg/m}^3$, pressure = $10^5 \text{ Pa}$
  • Initial state of water is $v_x=0 \text{ m/s}$, $v_y=-15 \text{ m/s}$, density = $1 \text{ kg/m}^3$, pressure = $10^5 \text{ Pa}$
  • Gravity is $g=-9.81 \text{ m/s}^2$
  • Boundary conditions are wall with perfect sliding
  • Box is $20 \text{ m} \times 15 \text{ m}$
  • Water is $10 \text{ m} \times 8 \text{ m}$
  • Mesh is $100x75 \text{ cells}$
2D RESULTS

- Fall of a volume of water on the ground in air
  - Video
2D RESULTS

Air is ejected by sliding between water and wall even when there is only one mixed cell left!
2D RESULTS

• Pressure field

$t=0.09\ s$

$t=0.132\ s$
2D RESULTS

- Shock wave in water interaction with a bubble of air
  - Shock wave at Mach 6
  - 200x100 cells

Water
\[
\rho = 1321.1 \text{kg m}^{-3} \\
\rho = 1.8381 \cdot 10^{10} \text{Pa} \\
u = 2113.68 \text{m s}^{-1}
\]

Air
\[
\rho = 1 \text{kg m}^{-3} \\
\rho = 10^5 \text{Pa} \\
u = 0 \text{m s}^{-1}
\]
2D RESULTS

- Geometry

[T=0]

[T=2.6µs]

[T=6.8µs]

[T=8.0µs]

[T=8.3µs]

[T=9.3µs]
2D RESULTS

- Pressure
CONCLUSION

• The method is locally conservative in mass, momentum and total energy, and allow sliding of materials by each others.
• Next studies on NIP aimed to improve accuracy in the mixed cells and a second order interface capturing.
• More physics, as surface tension or turbulence, will be added to this modelling.