# Composite Schemes

# for Conservation Laws

Richard Liska[1], Burton Wendroff[2]

[1]Faculty of Nuclear Sciences and Physical Engineering
Czech Technical University in Prague
Břehová 7, 115 19 Prague 1
Czech Republic

[2]Los Alamos National Laboratory
Los Alamos, NM 87544, U.S.A.

## Introduction

- finite difference methods for hyperbolic systems
- Lax-Friedrichs
- Godunov
- TVD, ENO, WENO, for systems need expansion in the eigenvectors of the gradient of the flux function

## 1D Schemes

- 1D system

$$U_t = f_x(U)$$

- Lax-Friedrichs ($L_F$) first half step

$$U_{i+1/2}^{n+1/2} = \frac{1}{2}[U_i^n + U_{i+1}^n] + \frac{\Delta t}{2\Delta x}[f(U_{i+1}^n) - f(U_i^n)]$$

  second half step is same with shifted indices

- two-step form of Lax-Wendroff ($L_W$) uses the same first half step and the second half step is

$$U_i^{n+1} = U_i^n + \frac{\Delta t}{\Delta x}[f(U_{i+1/2}^{n+1/2}) - f(U_{i-1/2}^{n+1/2})]$$

- $L_F$ is 1-st order accurate, highly diffusive
- $L_W$ is 2-nd order accurate, oscilatory for non-smooth solutions
- global composition of the $L_W$ and $L_F$ schemes by $k-1$ applications of $L_W$ followed by one application of $L_F$

$$S_k = L_F \circ L_W \circ \cdots \circ L_W$$

  so that

$$U^{n+k} = S_k U^n$$

  defines composite LWLFk scheme

## Shallow water flow over an obstacle

- system

$$\begin{aligned} h_t + (hu)_x &= 0 \\ (hu)_t + \left(hu^2 + g\frac{1}{2}h^2\right)_x + ghz_{0x} &= 0 \end{aligned}$$

- initial conditions

$$h(x,0) + z_0(x) = 1, \quad u(x,0) = u_0$$

- free boundary conditions

$$h_x(A_x,t) = 0, \quad u_x(A_x,t) = 0,$$

$$h_x(B_x,t) = 0, \quad u_x(B_x,t) = 0$$

- bottom profile

$$z_0(x) = \begin{cases} b_c\left(1 - \frac{x^2}{4}\right) & \text{for } -2 \le x \le 2 \\ 0 & \text{otherwise} \end{cases}$$

- schemes: LF, LW, LWLFn, LF with antidiffusion, LW with filter

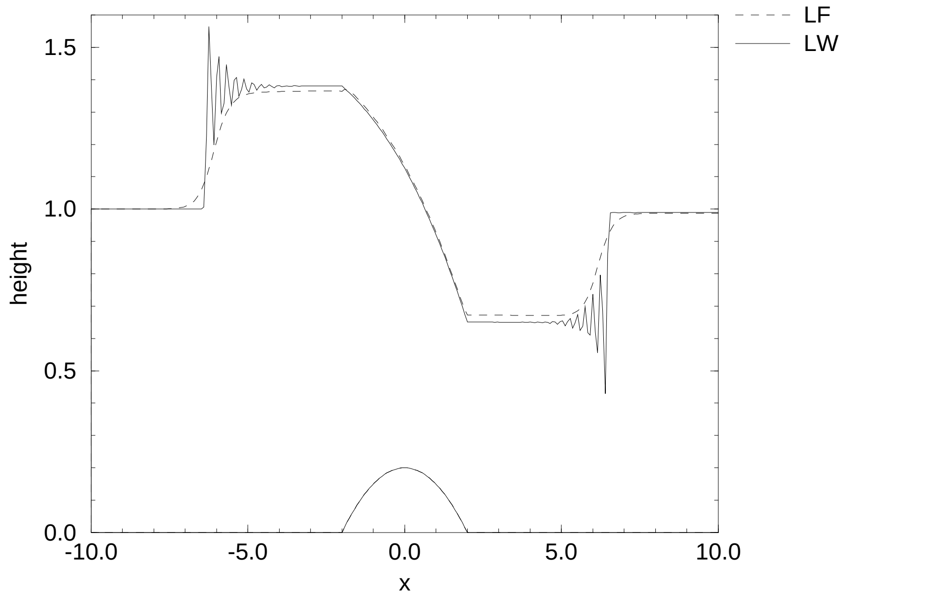


Figure 1: Comparison of heights for the shallow water problem with $b_c = 0.2, u_0 = 1$ at $t = 20$ calculated by Lax-Friedrichs and Lax-Wendroff two-step schemes using 250 points.

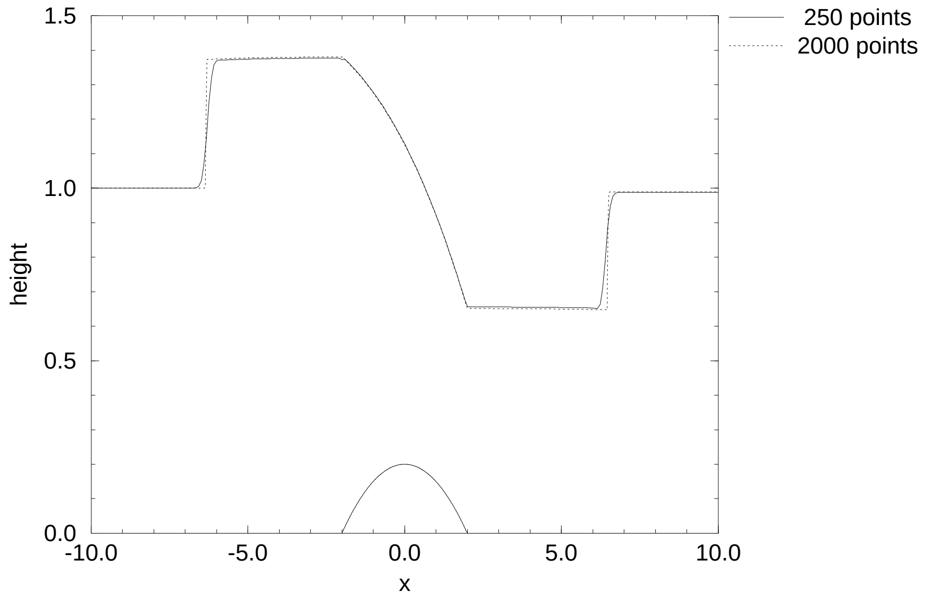


Figure 2: Comparison of heights for the shallow water problem with $b_c = 0.2, u_0 = 1$ at $t = 20$ calculated by the composite LWLF4 scheme with 250 and 2000 points.

## 2D composite scheme

- based on the observation of Boukadida and LeRoux (2D Godunov): to get cell averages on the dual grid from the averages on the primary grid one need only solve one-dimensional Riemann problems on the edges of the dual grid; exact 1D Riemann solutions are replaced by 1D LF approximation
- new version of Lax-Friedrichs without dimensional splitting for the system $U_t = f_x(U) + g_y(U)$

$$\begin{aligned} U^{n+1/2}_{i+1/2,j+1/2} &= \frac{1}{4}[U^n_{i,j} + U^n_{i+1,j} + U^n_{i,j+1} + U^n_{i+1,j+1}] \\ &+ \frac{\Delta t}{2\Delta x}[F_{i+1,j+1/2} - F_{i,j+1/2}] \\ &+ \frac{\Delta t}{2\Delta y}[G_{i+1/2,j+1} - G_{i+1/2,j}] \end{aligned}$$

with fluxes

$$\begin{aligned} F_{i+1,j+1/2} &= f\left(\frac{1}{2}[U^n_{i+1,j+1} + U^n_{i+1,j}] \right. \\ &\left. \qquad + \frac{\Delta t}{4\Delta y}[g(U^n_{i+1,j+1}) - g(U^n_{i+1,j})]\right), \\ G_{i+1/2,j+1} &= g\left(\frac{1}{2}[U^n_{i+1,j+1} + U^n_{i,j+1}] \right. \\ &\left. \qquad + \frac{\Delta t}{4\Delta x}[f(U^n_{i+1,j+1}) - f(U^n_{i,j+1})]\right) \end{aligned}$$

- for $U$ is scalar and $f(U) = aU, \quad g(U) = bU$ the scheme then is just transport projection
- second order accurate corrected Lax-Friedrichs (CF) scheme has corrector

$$
\begin{aligned}
U_{i,j}^{n+1} &= U_{i,j}^{n} \\
&+ \frac{\Delta t}{2\Delta x}[f(U_{i+1/2,j+1/2}^{n+1/2}) + f(U_{i+1/2,j-1/2}^{n+1/2}) \\
&\qquad - f(U_{i-1/2,j+1/2}^{n+1/2}) - f(U_{i-1/2,j-1/2}^{n+1/2})] \\
&+ \frac{\Delta t}{2\Delta y}[g(U_{i+1/2,j+1/2}^{n+1/2}) + g(U_{i-1/2,j+1/2}^{n+1/2}) \\
&\qquad - g(U_{i+1/2,j-1/2}^{n+1/2}) - g(U_{i-1/2,j-1/2}^{n+1/2})]
\end{aligned}
$$

- for scalar advection is an optimally stable that is, the stability condition is $\max(|\lambda|, |\mu|) \leq 1$, $\lambda = a\Delta t/\Delta x, \quad \mu = b\Delta t/\Delta y$
- the composite CFLFn consisting of $n-1$ applications of CF followed be one LF

$$LF \circ CF \circ \cdots \circ CF$$

## Gas dynamic tests

- Euler equations for an ideal gas in 2D

$$U_t + F(U)_x + G(U)_y = 0$$

where

$$U = \begin{pmatrix} \rho \\ \rho u \\ \rho v \\ \rho E \end{pmatrix}, \quad F = \begin{pmatrix} \rho u \\ \rho u^2 + p \\ \rho u v \\ u(\rho E + p) \end{pmatrix}, \quad G = \begin{pmatrix} \rho v \\ \rho u v \\ \rho v^2 + p \\ v(\rho E + p) \end{pmatrix},$$

$$E = e + 1/2(u^2 + v^2),$$

$$p = (\gamma - 1)\rho e$$

- Noh's infinite strength cylindrical shock problem, $\gamma = 5/3, \rho_0 = 1, p_0 = 0$ the initial velocities are directed toward the origin with magnitude 1

  the known solution is an infinite strength circularly symmetric shock reflecting from the origin; the density behind the shock is 16, the shock speed is 1/3 and ahead of the shock, that is for $\sqrt{x^2 + y^2} > t/3$, the density is $(1 + t/\sqrt{x^2 + y^2})$

- Riemann problems with initial conditions for $V = (p, \rho, u, v)$ constant in 4 quadrants configutations 4 (several shocks) and 6 (contact discontinuities) of Lax and Liu 1996

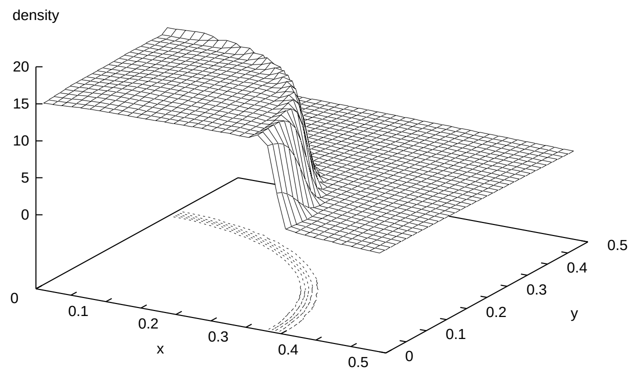


Figure 3: Surface and contour plot of density for the Noh problem at time $t = 1$ computed by CFLF4 scheme on a 75 by 75 grid.

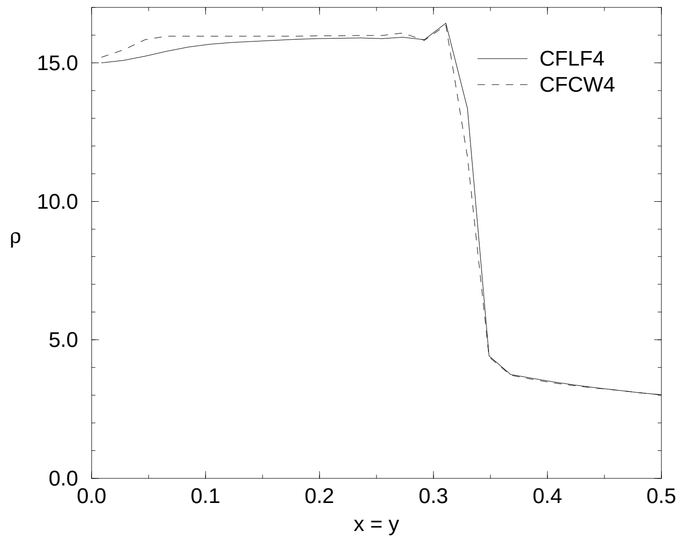


Figure 4: Density profile of the Noh problem at time $t = 1$ computed by CFLF4 scheme and CFCW4 2-nd order scheme on 75 by 75 grid on the diagonal line $x = y$.

- CFCW4 is CF composed with component-wise 2-nd order diffusive WENO scheme (no eigenvector decomposition)

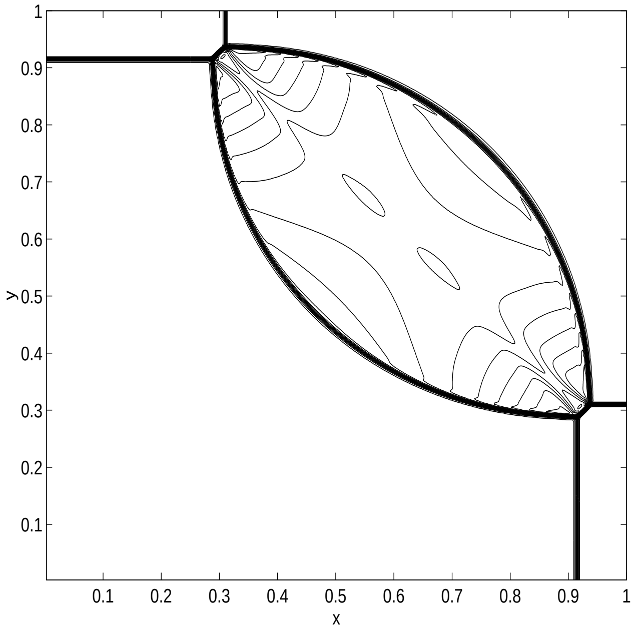


Figure 5: Contour plot of density for the 2D Riemann problem for an ideal gas with $\gamma = 1.4$ for configuration 4 done with the CFLF4 scheme at $t = 0.25$, 269 time steps, $\Delta x = \Delta y = 1/400$, 30 contours.

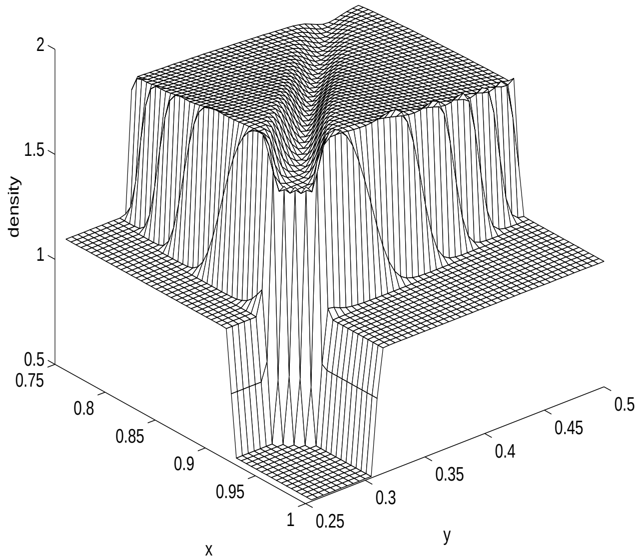


Figure 6: Surface plot of density in the region $(x, y) \in (0.75, 1) \times (0.25, 0.5)$ for the 2D Riemann problem for an ideal gas for configuration 4 with $\gamma = 1.4$ done with the CFLF4 scheme at $t = 0.25$, 550 time steps, $\Delta x = \Delta y = 1/800$

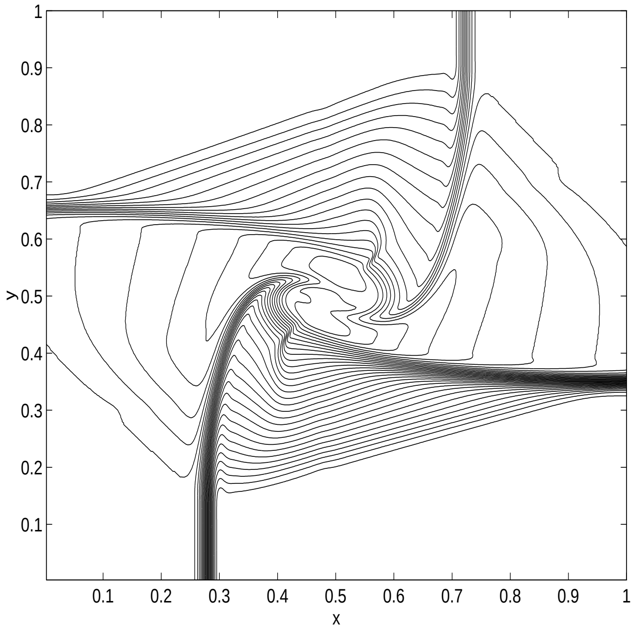


Figure 7: Contour plot of density for the 2D Riemann problem for an ideal gas with $\gamma = 1.4$ for configuration 6 done with the CFLF6 scheme at $t = 0.3$, 244 time steps, $\Delta x = \Delta y = 1/400$, 30 contours.

## Schemes in 3D

- natural generalization gives the new two-step LF 3-D scheme for scalar advection is positive and so optimally stable
- with 2-nd order corrector it is the two-step CF 3-D scheme for scalar advection is unconditionally unstable
- composite CFLFn is suboptimally stable (the whole cube $(-r_n, r_n)^3$ is inside the stability region)

| $n$ | $r_n$ |
|---|---|
| 2 | 0.9299 |
| 3 | 0.8979 |
| 4 | 0.8770 |
| 5 | 0.8602 |

## Conclusion

- composite schemes obtained by global composition of 2-nd order oscilatory scheme with a diffusive scheme
- no eigenvector decomposition
- no dimensional splitting
- composite schemes are
  - simple
  - fast
  - robust
  - optimally stable in 1D and 2D
  - and show very good resolution of both continuous and discontinuous solutions in 1D and 2D
- in 3D 2-nd order predictor corrector is unstable, however composite scheme is sub-optimally stable