The Stability of Boundary Conditions

Richard Liska[1], Stanly Steinberg[2]

[1]Faculty of Nuclear Sciences and Physical Engineering
Czech Technical University in Prague
Břehová 7, 115 19 Prague 1
Czech Republic
liska@siduri.fjfi.cvut.cz

[2]Department of Mathematics and Statistics
University of New Mexico
Albuquerque, New Mexico 87131
stanly@math.unm.edu

Initial-Boundary Value Problem

- one PDE

$$S(\partial_t, \partial_y, \partial_{\vec{x}}, \vec{\alpha})\, f(t, y, \vec{x}) = 0$$

- one boundary condition

$$B(\partial_t, \partial_y, \partial_{\vec{x}}, \vec{\alpha})\;\; f(t, y, \vec{x})|_{y=0} = 0$$

- trial function

$$f(t, y, \vec{x}) = e^{\lambda\, t + \eta\, y + \imath\, \vec{\xi} \cdot \vec{x}}\, v(\lambda, \eta, \imath\, \vec{\xi})$$

- existence of unstable solution

$$\begin{aligned} &\exists \lambda \in \mathcal{C}\, \exists \eta \in \mathcal{C}\, \exists \vec{\xi} \in \mathcal{R}^d \\ &\quad \Big\{ \Re\lambda > 0 \wedge \Re\vec{\eta} < 0 \wedge S(\lambda, \eta, \vec{\xi}, \vec{\alpha}) = 0 \\ &\qquad \wedge B(\lambda, \eta, \vec{\xi}, \vec{\alpha}) = 0 \Big\} \end{aligned}$$

- can be generalized to system of PDEs with BCs

Example - Initial-Boundary Value Problem

- heat equation with Robin BCs

$$\frac{\partial f}{\partial t} = \alpha \frac{\partial^2 f}{\partial x^2} + \beta \frac{\partial^2 f}{\partial^2 y}, \quad \left(a \frac{\partial f}{\partial y} + b\, f\right)_{y=0} = 0$$

- characteristic polynomial

$$C(\lambda,\, \eta) = \lambda + \alpha\, \xi^2 - \beta\, \eta^2$$

$$B(\lambda,\, \eta) = a\, \eta + b\,.$$

- exists unstable solution of IBVP

$$\begin{aligned} &\exists\lambda \in \mathcal{C}\, \exists\eta \in \mathcal{C}\, \exists\xi \in \mathcal{R} \\ &\quad \{\Re\lambda > 0 \wedge \Re\eta < 0 \wedge \\ &\qquad \lambda + \alpha\, \xi^2 - \beta\, \eta^2 = 0 \wedge a\, \eta + b = 0\} \end{aligned}$$

- instability region

$$\alpha \geq 0 \wedge \beta > 0 \wedge (a\, b > 0 \vee a = b = 0)$$

Initial-Boundary Value Problem for System of PDEs

- system of N PDEs ($\mathbf{S}$ is $N \times N$ matrix)

$$\mathbf{S}(\partial_t, \partial_y, \partial_{\vec{x}}, \vec{\alpha})\, \vec{f}(t, y, \vec{x}) = 0$$

- M boundary conditions ($\mathbf{B}$ is $M \times N$ matrix)

$$\mathbf{B}(\partial_t, \partial_y, \partial_{\vec{x}}, \vec{\alpha})\;\, \vec{f}(t, y, \vec{x})\Big|_{y=0} = 0$$

- trial function

$$\vec{f}(t, y, \vec{x}) = e^{\lambda\, t + \eta\, y + \imath\, \vec{\xi}\cdot\vec{x}}\, \vec{v}(\lambda, \eta, \imath\, \vec{\xi})$$

- $\vec{f}$ can be a non-trivial solution of the PDEs if

$$C(\lambda, \eta, \vec{\xi}, \vec{\alpha}) = \det \mathbf{S}(\lambda, \eta, \imath\, \vec{\xi}, \vec{\alpha}) = 0$$

$$\mathbf{S}(\lambda, \eta, \imath\, \vec{\xi}, \vec{\alpha})\, \vec{v}(\lambda, \eta, \imath\, \vec{\xi}, \vec{\alpha}) = 0$$

- K roots $\eta_k = \eta_k(\lambda, \vec{\xi}, \vec{\alpha})$ of $C(\lambda, \eta, \vec{\xi}, \vec{\alpha}) = 0$ with corresponding null vectors $\vec{v}_k(\lambda, \eta_k, \vec{\xi}, \vec{\alpha})$ for which

$$\Re\, \lambda > 0\,, \quad \Re\, \eta < 0\,, \quad \vec{\xi} \in \mathcal{R}^d$$

give unstable solutions, roots η_k are assumed to be simple

- general unstable solution

$$\vec{f}(t,x,\vec{y}) = \sum_{k=1}^{K} c_k \, e^{\lambda\, t + \eta_k(\lambda, \vec{\xi}, \vec{\alpha})\, y + \imath\, \vec{\xi}\, \vec{x}} \, \vec{v}_k(\lambda,\, \eta_k,\, \vec{\xi},\, \vec{\alpha})$$

  is used as trial function for BCs

$$\sum_{k=1}^{K} c_k \, \mathbf{B}(\lambda,\, \eta_k(\lambda,\, \vec{\xi},\, \vec{\alpha}),\, \imath\, \vec{\xi})\, \vec{v}_k(\lambda,\, \eta_k, \vec{\xi},\, \vec{\alpha}) = 0$$

- this can be written as ($\mathbf{T}$ is $M \times K$ matrix)

$$\mathbf{T}\, \vec{c} = 0$$

- if $M < K$ then it has always a non-trivial solution $\vec{c} \neq 0$ and the BCs are unstable
- if $M > K$ there are too many BCs
- assume $M = K$
- existence of unstable solution

$$\begin{aligned}
&\exists \lambda \in \mathcal{C}\ \exists \vec{\eta} \in \mathcal{C}^K\ \exists \vec{\xi} \in \mathcal{R}^d\ \exists \vec{v}_1 \in \mathcal{R}^N\ \cdots \exists \vec{v}_K \in \mathcal{R}^N \\
&\quad \Big\{ \Re\lambda > 0 \wedge \Re\vec{\eta} < 0 \wedge \mathrm{Distinct}(\vec{\eta}) \wedge \\
&\qquad \bigwedge_{k=1}^{K} \{ C(\lambda,\, \eta_k,\, \vec{\xi},\, \vec{\alpha}) = 0 \wedge \\
&\qquad \mathbf{S}(\lambda,\, \eta_k,\, \imath\vec{\xi},\, \vec{\alpha})\, \vec{v}_k = 0 \wedge \vec{v}_k \neq 0 \} \wedge \\
&\qquad \det(\mathbf{T}(\lambda,\, \vec{\eta},\, \vec{\xi},\, \vec{\alpha})) = 0 \Big\}\ ,
\end{aligned}$$

  where

$$\mathrm{Distinct}(\vec{\eta}) = \bigwedge_{i,j=1}^{K} i \neq j \Rightarrow \eta_i \neq \eta_j$$

Example - Initial-Boundary Value Problem for System of PDEs

- hyperbolic system

$$u_t + u_y - v_x = 0$$
$$v_t - v_y - u_x = 0$$

- boundary condition

$$(u + a\,v)_{y=0} = 0$$

- matrix $\mathbf{S}$

$$\mathbf{S}(\lambda,\, \eta,\, \imath\,\xi) = \begin{pmatrix} \lambda + \eta & -\imath\,\xi \\ -\imath\,\xi & \lambda - \eta \end{pmatrix}$$

- matrix $\mathbf{B}$

$$\mathbf{B}(\lambda,\, \eta,\, \imath\,\xi) = (1,\, a)$$

- characteristic polynomial

$$C(\lambda,\, \eta,\, \xi) = \lambda^2 - \eta^2 + \xi^2$$

has one root η with $\Re\,\eta < 0$ so $K = 1$

- null vectors

$$\vec{v}_1 = \begin{pmatrix} \lambda - \eta \\ \imath\,\xi \end{pmatrix}, \qquad \vec{v}_2 = \begin{pmatrix} \imath\,\xi \\ \lambda + \eta \end{pmatrix}$$

- matrix $\mathbf{T}$ is a scalar $T = \mathbf{B}\vec{v}_k$
- for $\vec{v}_1$, T cannot be zero, so $\vec{v}_1$ cannot generate unstable solution
- for $\vec{v}_2$ we have $T = \imath\,\xi + a\,(\lambda + \eta)$ and the existence of unstable solutions is implied by

$$\begin{aligned} &\exists\lambda \in \mathcal{C}\ \exists\eta \in \mathcal{C}\ \exists\xi \in \mathcal{R} \\ &\quad \{\Re\lambda > 0 \wedge \Re\eta < 0 \wedge \lambda^2 - \eta^2 + \xi^2 = 0 \wedge \\ &\qquad (\xi \neq 0 \vee \lambda + \eta \neq 0) \wedge \\ &\qquad \imath\,\xi + a\,(\lambda + \eta) = 0\} \end{aligned}$$

- for $a \in \mathcal{R}$ the result is false, so the BCs are stable
- with equational constraints QEPCAD finishes in 1 s of CPU time with 4 MB of RAM
- without equational constraints QEPCAD fails after more than 5 hours of CPU time with 64 MB of RAM

Semidiscrete IBVP

- semidiscretization

$$f_{m,\vec{j}}(t) \approx f(t, m\Delta y, \vec{j} * \Delta\vec{x})$$

- semidiscrete initial boundary value problem, one PDE, one BC

$$S(\partial_t, \delta_y, \delta_{\vec{x}}, \vec{\alpha}) f_{m,\vec{j}} = 0\,, \quad B(\partial_t, \delta_y, \delta_{\vec{x}}, \vec{\alpha}) f_{0,\vec{j}} = 0$$

- trial function

$$f_{m,\vec{j}}(t) = e^{\lambda t}\, w^m\, \vec{z}^{\vec{j}}\,, \quad |z_k| = 1\,, \quad k = 1, \cdots, d$$

- existence of unstable solution

$$\begin{aligned} &\exists \lambda \in \mathcal{C}\, \exists w \in \mathcal{C}\, \exists \vec{z} \in \mathcal{C}^d \\ &\quad \{\Re\lambda > 0 \wedge |w| < 1 \wedge |\vec{z}| = 1 \wedge \\ &\qquad \wedge S(\lambda, w, \vec{z}, \vec{\alpha}) = 0 \wedge B(\lambda, w, \vec{z}, \vec{\alpha}) = 0\} \end{aligned}$$

- can be generalized to system of PDEs with BCs

Example - Semidiscrete IBVP

- fourth-order semidiscretization, Carpenter, Gottlieb and Abarbanel (1993) for 1 D transport equation

$$f'_{m+1} + 4f'_m + f'_{m-1} + \frac{3(f_{m+1} - f_{m-1})}{\Delta y} = 0$$

- combined first- and second-order BC

$$\begin{aligned} &(3 + 4\beta)\, f'_1 + (1 + 2\beta) f'_2 \\ &\quad + \frac{1}{\Delta y}\left[-2(1+\beta) f_1 + 3(1 + 2\beta) f_2\right] = 0 \end{aligned}$$

- characteristic polynomial and symbol of BC

$$\begin{aligned} C(\Lambda, w) &= (1 + 4w + w^2)\Lambda + 3(w^2 - 1), \quad \Lambda = \lambda \Delta y \\ B(\Lambda, w) &= [3 + 4\beta + (1 + 2\beta) w]\Lambda \\ &\qquad -2(1 + \beta) + 3(1 + 2\beta) w \end{aligned}$$

- existence of unstable solution

$$\begin{gathered} \exists \Lambda \in \mathcal{C} \exists w \in \mathcal{C} \{\Re\Lambda > 0 \wedge |w| < 1 \wedge \\ C(\Lambda, w) = 0 \wedge B(\Lambda, w) = 0\} \end{gathered}$$

- instability region

$$4\beta + 1 < 0 \wedge 2\beta^2 - 2\beta - 1 < 0$$

Discrete IBVP

- discretization

$$f^n_{m,\vec{j}} \approx f(n\Delta t, m\Delta y, \vec{j} * \Delta\vec{x})$$

- discrete initial boundary value problem, one difference scheme, one BC

$$S(\delta_t, \delta_y, \delta_{\vec{x}}, \vec{\alpha}) f^n_{m,\vec{j}} = 0\,, \quad B(\delta_t, \delta_y, \delta_{\vec{x}}, \vec{\alpha}) f^n_{0,\vec{j}} = 0$$

- trial function

$$f^n_{m,\vec{j}} = s^n\, w^m\, \vec{z}^{\vec{j}}\,, \quad |z_k| = 1\,, \quad k = 1, \cdots, d$$

- existence of unstable solution

$$\begin{aligned} &\exists s \in \mathcal{C}\, \exists w \in \mathcal{C}\, \exists \vec{z} \in \mathcal{C}^d \\ &\qquad \{|s| > 1 \wedge |w| < 1 \wedge |\vec{z}| = 1 \wedge \\ &\qquad\qquad \wedge S(\lambda, w, \vec{z}, \vec{\alpha}) = 0 \wedge B(\lambda, w, \vec{z}, \vec{\alpha}) = 0\} \end{aligned}$$

- can be generalized to system of difference schemes with BCs