Analysis and Computation with Multi-layer Fluid Flows

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Two-layer Flow - Denotation

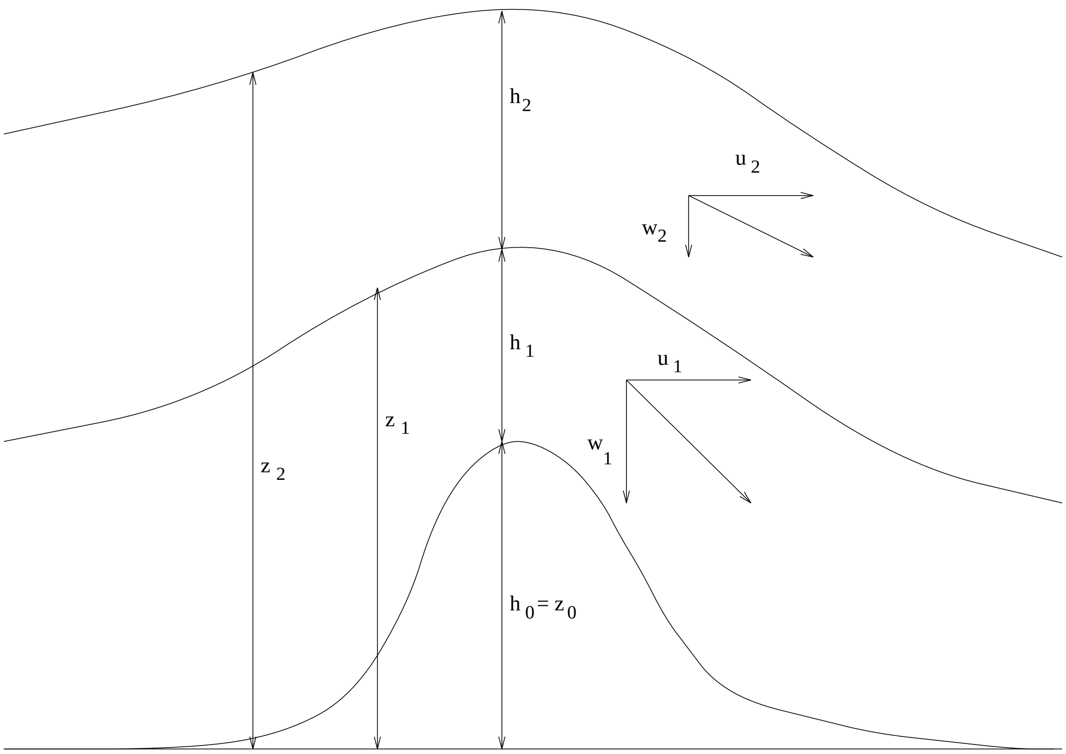

Figure 1: Denotation of thicknesses, heights and velocities for two-layer models.

- bottom profile $h_0(x) = z_0(x)$
- thicknesses $h_1(x,t), h_2(x,t)$
- heights $z_1(x,t) = h_0 + h_1, z_2(x,t) = h_0 + h_1 + h_2$
- horizontal velocities $u_1(x,t), u_2(x,t)$
- vertical velocities $w_1(x,t), w_2(x,t)$
- densities ρ_1, ρ_2 are constant

Overview

- motivation
- consider two-layer models in 2D
- Euler equations
- verticaly avegared two-layer models
  - shallow water equation
  - Green-Naghdi equations
- well-posedness analysis of the models
- discretization of the models
- numerical results
- conclusion

Motivation

- vertically averaged 1D or 2D models have obvious computational advantage over 2D or 3D Euler equations models
- one-layer models give a good approximation in many cases to full dimensional models
- multi-layer models are natural generalization of single-layer models, however several difficulties are associated with these models
- ocean and atmosphere modelling

Euler Equations

- 2D Euler equations for two layers, $i = 1, 2$

$$
\begin{aligned}
u_{ix} + w_{iz} &= 0 && (1) \\
\rho_i(u_{it} + u_i u_{ix} + w_i u_{iz}) &= -p_{ix} && (2) \\
\rho_i(w_{it} + u_i w_{ix} + w_i w_{iz}) &= -p_{iz} - g\rho_i && (3)
\end{aligned}
$$

- reduced dimension models
  - assumed form of dependence of velocities u_i, w_i on z
  - (1) and BC result into mass conservation equations
  - (3) is integrated over (z, z_i) to obtain $p_i(z)$
  - $p_i(z)$ is substituted into (2) which is then integrated over (z_{i-1}, z_i) to obtain momentum conservation equations

Boundary Conditions for Euler Equations

- define

$$d_i f = f_t + u_i f_x$$

- at $z = z_0$

$$\mathrm{d}_1 z_0 = w_1$$

- at $z = z_1$

$$\mathrm{d}_1 z_1 = w_1$$

$$\mathrm{d}_2 z_1 = w_2$$

$$p_1 = p_2 \quad \text{continuous pressure}$$

- at $z = z_2$
  free surface condition

$$\mathrm{d}_2 z_2 = w_2, \quad \text{and} \quad p_2 = 0$$

  rigid lid condition

$$z_2 = \text{constant}, \qquad w_2 = 0$$

Shallow Water Free Surface

- assumptions

$$u_{iz} \equiv 0, \quad i = 1, 2$$

$$p_{iz} + g\rho_i = 0$$

- mass conservatin equations are obtained from BCs and incompressibility condition

$$h_{it} + (h_i u_i)_x = 0, \quad i = 1, 2$$

- pressures

$$p_2(z) = -\rho_2 g(z - z_2)$$

$$p_1(z) = \rho_2 g h_2 - \rho_1 g(z - z_1)$$

- momentum equations are obtained by

$$\int_{z_{i-1}}^{z_i} [\rho_i(u_{it} + u_i u_{ix}) + p_{ix}] \mathrm{d}\, z = 0, \quad i = 1, 2$$

- momentum equations in terms of momenta

$$(h_1 u_1)_t + \left(h_1 u_1^2 + \frac{1}{2} g h_1^2\right)_x + \frac{\rho_2}{\rho_1} g h_1 h_{2x} + g h_1 z_{0x} = 0$$

$$(h_2 u_2)_t + \left(h_2 u_2^2 + \frac{1}{2} g h_2^2\right)_x + g h_2 h_{1x} + g h_2 z_{0x} = 0$$

Two-layer Green-Naghdi

- assume u_i independent of z and w_i linear in z

$$u_{iz} \equiv 0, \quad i = 1, 2$$

$$w_i = a_i(x,t) - z u_{ix}$$

- from one BC and incompressibility condition

$$w_i = \mathrm{d}_i z_{i-1} - (z - z_{i-1}) u_{ix}$$

- mass conservatin equations are obtained from the other BC

$$h_{it} + (h_i u_i)_x = 0, \quad i = 1, 2$$

- pressures

$$p_1(z) = q - \rho_1 \int_{z_1}^{z} (w_{1t} + u_1 w_{1x} + w_1 w_{1z} + g) \mathrm{d}\, z$$

$$p_2(z) = -\rho_2 \int_{z_2}^{z} (w_{2t} + u_2 w_{2x} + w_2 w_{2z} + g) \mathrm{d}\, z$$

- momentum equations are obtained by

$$\int_{z_{i-1}}^{z_i} [\rho_i (u_{it} + u_i u_{ix} + w_i u_{iz}) + p_{ix}] \mathrm{d}\, z = 0, \quad i = 1, 2$$

- momentum equations after rearrangement

$$(z_i - z_{i-1})\mathrm{d}_i u_i + (z_i - z_{i-1}) g z_{ix} - A_{ix} - B_i z_{(i-1)x} = -1/\rho_i (z_i - z_{i-1})\mathcal{P}_{ix}$$

where

$$A_i = (z_i - z_{i-1})^2 \mathrm{d}_i \left[\frac{1}{3}(z_i - z_{i-1}) u_{ix} - \frac{1}{2}\mathrm{d}_i z_{i-1}\right]$$

$$B_i = (z_i - z_{i-1}) \mathrm{d}_i \left[\frac{1}{2}(z_i - z_{i-1}) u_{ix} - \mathrm{d}_i z_{i-1}\right]$$

and $\mathcal{P}_i$ is the pressure at the interface

- for free surface $\mathcal{P}_2 = 0$ and

$$\mathcal{P}_1 = \rho_2 \left[q_2(z_2) - q_2(z_1) + g(z_2 - z_1)\right]$$

where

$$q_i(s) = s\mathrm{d}_i^2 z_{i-1} - \frac{\frac{s^2}{2} - s z_{i-1}}{z_i - z_{i-1}} \mathrm{d}_i \left[(z_i - z_{i-1}) u_{ix}\right]$$

Two-layer Green-Naghdi

- expanded PDEs for free surface

$$h_{1t} + h_{1x}u_1 + u_{1x}h_1 = 0$$

$$h_{2t} + h_{2x}u_2 + u_{2x}h_2 = 0$$

$$\begin{aligned}
&6\rho_2 g h_{2x} h_1 + 3\rho_2 h_1 \Big(4h_{2x}h_{1tx}u_2 + 2h_{2x}h_{1tt} + 2h_{2x}h_{1xx}u_2^2 + 2h_{2x}h_{1x}u_{2t} \\
&\quad + 2h_{2x}h_{1x}u_{2x}u_2 + 2h_{2x}h_{0xx}u_2^2 + 2h_{2x}h_{0x}u_{2t} + 2h_{2x}h_{0x}u_{2x}u_2 - 2h_{2x}u_{2tx}h_2 \\
&\quad - 2h_{2x}u_{2xx}h_2u_2 + 2h_{2x}u_{2x}^2h_2 + 4h_{1txx}h_2u_2 + 4h_{1tx}u_{2x}h_2 + 2h_{1ttx}h_2 \\
&\quad + 2h_{1xxx}h_2u_2^2 + 2h_{1xx}u_{2t}h_2 + 6h_{1xx}u_{2x}h_2u_2 + 2h_{1x}u_{2tx}h_2 + 2h_{1x}u_{2xx}h_2u_2 \\
&\quad + 2h_{1x}u_{2x}^2h_2 + 2h_{0xxx}h_2u_2^2 + 2h_{0xx}u_{2t}h_2 + 6h_{0xx}u_{2x}h_2u_2 + 2h_{0x}u_{2tx}h_2 \\
&\quad + 2h_{0x}u_{2xx}h_2u_2 + 2h_{0x}u_{2x}^2h_2 - u_{2txx}h_2^2 - u_{2xxx}h_2^2u_2 + u_{2xx}u_{2x}h_2^2\Big) \\
&\quad + 6\rho_1 g h_1 \left(h_{1x} + h_{0x}\right) + \rho_1 h_1 \Big(6h_{1x}h_{0xx}u_1^2 + 6h_{1x}h_{0x}u_{1t} + 6h_{1x}h_{0x}u_{1x}u_1 \\
&\quad - 6h_{1x}u_{1tx}h_1 - 6h_{1x}u_{1xx}h_1u_1 + 6h_{1x}u_{1x}^2h_1 + 3h_{0xxx}h_1u_1^2 + 6h_{0xx}h_{0x}u_1^2 \\
&\quad + 3h_{0xx}u_{1t}h_1 + 9h_{0xx}u_{1x}h_1u_1 + 6h_{0x}^2u_{1t} + 6h_{0x}^2u_{1x}u_1 + 6h_{0x}u_{1x}^2h_1 \\
&\quad - 2u_{1txx}h_1^2 + 6u_{1t} - 2u_{1xxx}h_1^2u_1 + 2u_{1xx}u_{1x}h_1^2 + 6u_{1x}u_1\Big) = 0
\end{aligned}$$

$$\begin{aligned}
&6\rho_2 g h_2 \left(h_{2x} + h_{1x} + h_{0x}\right) + \rho_2 h_2 \Big(12h_{2x}h_{1tx}u_2 + 6h_{2x}h_{1tt} + 6h_{2x}h_{1xx}u_2^2 \\
&\quad + 6h_{2x}h_{1x}u_{2t} + 6h_{2x}h_{1x}u_{2x}u_2 + 6h_{2x}h_{0xx}u_2^2 + 6h_{2x}h_{0x}u_{2t} + 6h_{2x}h_{0x}u_{2x}u_2 \\
&\quad - 6h_{2x}u_{2tx}h_2 - 6h_{2x}u_{2xx}h_2u_2 + 6h_{2x}u_{2x}^2h_2 + 6h_{1txx}h_2u_2 + 12h_{1tx}h_{1x}u_2 \\
&\quad + 12h_{1tx}h_{0x}u_2 + 6h_{1tx}u_{2x}h_2 + 3h_{1ttx}h_2 + 6h_{1tt}h_{1x} + 6h_{1tt}h_{0x} \\
&\quad + 3h_{1xxx}h_2u_2^2 + 6h_{1xx}h_{1x}u_2^2 + 6h_{1xx}h_{0x}u_2^2 + 3h_{1xx}u_{2t}h_2 + 9h_{1xx}u_{2x}h_2u_2 \\
&\quad + 6h_{1x}^2u_{2t} + 6h_{1x}^2u_{2x}u_2 + 6h_{1x}h_{0xx}u_2^2 + 12h_{1x}h_{0x}u_{2t} + 12h_{1x}h_{0x}u_{2x}u_2 \\
&\quad + 6h_{1x}u_{2x}^2h_2 + 3h_{0xxx}h_2u_2^2 + 6h_{0xx}h_{0x}u_2^2 + 3h_{0xx}u_{2t}h_2 + 9h_{0xx}u_{2x}h_2u_2 \\
&\quad + 6h_{0x}^2u_{2t} + 6h_{0x}^2u_{2x}u_2 + 6h_{0x}u_{2x}^2h_2 - 2u_{2txx}h_2^2 + 6u_{2t} \\
&\quad - 2u_{2xxx}h_2^2u_2 + 2u_{2xx}u_{2x}h_2^2 + 6u_{2x}u_2\Big) = 0
\end{aligned}$$

Method of Well-posedness Analysis

- nonlinear system of PDEs, $\mathbf{u}(x,t)$

$$\mathbf{E}(\mathbf{u}) = 0$$

- linearization

$$\mathbf{u} = \bar{\mathbf{v}} + \epsilon\tilde{\mathbf{v}}$$

- coefficients of ϵ give a linear system

$$\mathbf{L}\,\tilde{\mathbf{v}} = 0$$

- dispersion analysis of the linear system

$$\frac{\partial^{n+m}\tilde{\mathbf{v}}}{\partial\, t^n\, \partial\, x^m} \to (i\omega)^n (ik)^m \hat{\mathbf{v}}$$

  give

$$\hat{\mathbf{L}}(\omega, k)\,\hat{\mathbf{v}} = 0,$$

- characteristic polynomial ($\lambda = \omega/k$)

$$P(\lambda, k) = \det\,\hat{\mathbf{L}}(\lambda k, k)$$

- roots λ_j of $P(\lambda, k) = 0$ in the limit $|k| \to \infty$ determine well-posedness

- $P_1(\lambda, k)$ obtained from $P(\lambda, k)$ by keeping only highest degree in k terms in each coefficient of powers of λ
- in most cases $P_1(\lambda, k) = k^m P_2(\lambda)$, then the system is ill-posed iff exists a root λ_j of $P_2(\lambda) = 0$ with non-zero imaginary part
- implemented in computer algebra system REDUCE

Results of Well-posedness Analysis

- two-layer models
- shalow water, free surface

$$\begin{aligned} P_1(\lambda) &= (\lambda+u_1)^2(\lambda+u_2)^2 - gh_1(\lambda+u_2)^2 \\ &\quad -gh_2(\lambda+u_1)^2 + g^2h_1h_2(1-\rho_2/\rho_1) \end{aligned}$$

  – case $\rho_1 = \rho_2$ is ill-posed for

$$0 < (u_1 - u_2)^2 < (H_1^{1/3} + H_2^{1/3})^3$$

  where $H_i = gh_i$

  – for $\rho_1 < \rho_2$ is ill-posed

  – for $\rho_1 > \rho_2$ there exists $\epsilon > 0$ such that for $|u_2 - u_1| < \epsilon$ the problem is well-posed

  – for $H_1 = H_2 = H$ is ill-posed for

$$2[H(1-\sqrt{R})]^{1/2} < |u_1 - u_2| < 2[H(1+\sqrt{R})]^{1/2}$$

- shallow water, rigid lid is well-posed for

$$(u_1 - u_2)^2 \le g(\rho_1 - \rho_2)\left(\frac{h_1}{\rho_1} + \frac{h_2}{\rho_2}\right)$$

- Green-Naghdi, free surface

$$P_1(\lambda) = (\lambda + u_2)^2 Q(\lambda)$$

where $Q(\lambda)$ is quadratic in λ and its discriminant is

$$D = -48(u_1 - u_2)^2 h_1 h_2 \rho_1 \rho_2$$

so it is unconditionaly ill-posed

- Green-Naghdi, rigid lid is also unconditionaly ill-posed
- Green-Naghdi, rigid lid with fourth order dissipation is well-posed

Other Approximations for Vertical Averaging

- other approximations have regions of well-posedness and regions of ill-posedness
- piecewise quadratic w
- integrations with weight
- point values instead of integral

Shallow Water Discretization

- system is hyperbolic in some region of state space and ill-posed in the rest
- wave speeds do not have simple expression
- cannot use methods employing eigenvector decomposition
- system is not in conservation form
- only 3 conserved quantities are known, mass in both layers and total momentum, so we have 3 shock jump conditions and probably universaly correct fourth one does not exist
- first order balance laws

$$\mathbf{w}_t + \mathbf{f}_x(\mathbf{w}) + \mathbf{G}(\mathbf{w})\mathbf{w}_x + g(\mathbf{w}, x) = 0$$

- predictor step

$$\begin{aligned}
\mathbf{w}_{j+1/2}^{n+1/2} = {} & \frac{1}{2}\left(\mathbf{w}_j^n + \mathbf{w}_{j+1}^n\right) - \frac{\Delta t}{2\Delta x}\left[\mathbf{f}(\mathbf{w}_{j+1}^n) - \mathbf{f}(\mathbf{w}_j^n)\right] \\
& -\frac{\Delta t}{2\Delta x}\mathbf{G}\left(\frac{1}{2}(\mathbf{w}_j^n + \mathbf{w}_{j+1}^n)\right)\left[\mathbf{w}_{j+1}^n - \mathbf{w}_j^n\right] \\
& -\frac{\Delta t}{2} g\left(\frac{1}{2}(\mathbf{w}_j^n + \mathbf{w}_{j+1}^n), x_{j+1/2}\right)
\end{aligned}$$

- two-step form of Lax-Friedrichs (LF) is the predictor above folloved by the corrector step

$$
\begin{aligned}
\mathbf{w}_j^{n+1} \;=\; & \frac{1}{2}\left(\mathbf{w}_{j+1/2}^{n+1/2} + \mathbf{w}_{j-1/2}^{n+1/2}\right) \\
& -\frac{\Delta t}{2\Delta x}\left[\mathbf{f}(\mathbf{w}_{j+1/2}^{n+1/2}) - \mathbf{f}(\mathbf{w}_{j-1/2}^{n+1/2})\right] \\
& -\frac{\Delta t}{2\Delta x}\mathbf{G}\left(\frac{1}{2}(\mathbf{w}_{j+1/2}^{n+1/2} + \mathbf{w}_{j-1/2}^{n+1/2})\right)\left[\mathbf{w}_{j+1/2}^{n+1/2} - \mathbf{w}_{j-1/2}^{n+1/2}\right] \\
& -\frac{\Delta t}{2} g\left(\frac{1}{2}(\mathbf{w}_{j+1/2}^{n+1/2} + \mathbf{w}_{j-1/2}^{n+1/2}), x_j\right)
\end{aligned}
$$

- the LF scheme with antidiffusion applied every n-th time step is denoted by LFADn
- two-step form of Lax-Wendroff (LW) is the same predictor followed by the corrector step

$$
\begin{aligned}
\mathbf{w}_j^{n+1} \;=\; & \mathbf{w}_j^{n} - \frac{\Delta t}{\Delta x}\left[\mathbf{f}(\mathbf{w}_{j+1/2}^{n+1/2}) - \mathbf{f}(\mathbf{w}_{j-1/2}^{n+1/2})\right] \\
& -\frac{\Delta t}{\Delta x}\mathbf{G}\left(\frac{1}{2}(\mathbf{w}_{j+1/2}^{n+1/2} + \mathbf{w}_{j-1/2}^{n+1/2})\right)\left[\mathbf{w}_{j+1/2}^{n+1/2} - \mathbf{w}_{j-1/2}^{n+1/2}\right] \\
& -\Delta t g\left(\frac{1}{2}(\mathbf{w}_{j+1/2}^{n+1/2} + \mathbf{w}_{j-1/2}^{n+1/2}), x_j\right)
\end{aligned}
$$

- composite scheme is got by replacing every n-th LW step by a LF step, we denote it LWLFn
- composite schemes work fine also on other problems, also in 2D

- Crank-Nicolson (CN)

$$\mathbf{w}_j^{n+1} = \mathbf{w}_j^n + \frac{1}{2}\Delta t[\mathbf{F}(\mathbf{w}^{n+1}) + \mathbf{F}(\mathbf{w}^n],$$

  is non-linear and is solved by Newton's method using Jacobian of $\mathbf{F}$, usually one iteratin is enough, it is implicit scheme, code generation used to produce the numerical code

- filter removing oscilations

- composit scheme CNLFnFm, every n-th CN step is replaced by LF and filter is applied every m-th step

One-layer Green-Naghdi Discretization

- time discretization

$$v_t \approx \frac{v^{n+1} - v^n}{\Delta t}$$

  all remaining term are discretized on implicit time level

- space discretization

$$\begin{aligned} v_x &\approx \frac{-v_{j+2} + 8v_{j+1} - 8v_{j-1} + v_{j-2}}{12\Delta x}, \\ v_{xx} &\approx \frac{-v_{j+2} + 16v_{j+1} - 30v_j + 16v_{j-1} - v_{j-2}}{12\Delta^2 x}, \\ v_{xxx} &\approx \frac{v_{j+2} - 2v_{j+1} + 2v_{j-1} - v_{j-2}}{2\Delta^3 x}, \end{aligned}$$

- resulting non-linear scheme is solved by Newton's method which results in iterative method, on each iteration step an implicit linear difference scheme has to be solved, quite complicated Jacobians, we call it backward Euler Newton (BEN) scheme

- code generation used to create the numerical code which has for this case about 90 kB, 1600 lines

Two-layer Green-Naghdi Discretization

- second time derivatives are eliminated by using mass conservation equation
- BEN scheme as done for one-layer Green-Naghdi resulted in more than 550 kB or 8000 lines of numerical code, Jacobians are huge
- BEN is too slow
- ad-hoc time discretization, in products not containing time derivative the term containing the unique highest order spatial derivative is taken on implicit level, for other products we use splitting as e.g.

  $$ab \approx (a^n b^{n+1} + a^{n+1} b^n)/2$$

- space discretization is same as as in one-layer case
- code generatin tools generate about 130 kB, 2500 lines of numerical code for solving this linear implicit scheme

Initial and Boundary Conditions for Two-layer Models

- initial conditions

$$z_i(x,0) = z_i^0, \quad u_i(x,0) = u_i^0, \quad i = 1,2$$

- solved on $x \in (A_x, B_x)$, free boundary conditions

$$\begin{aligned} z_{ix}(A_x,t) &= 0, \qquad u_{ix}(A_x,t) = 0, \\ z_{ix}(B_x,t) &= 0, \qquad u_{ix}(B_x,t) = 0, \quad i = 1,2 \end{aligned}$$

- three different bottom profiles

$$\begin{aligned} z_0(x) &= \begin{cases} b_c\left(1 - \frac{x^2}{a^2}\right) & \text{for } -a \le x \le a \\ 0 & \text{otherwise} \end{cases} \\ z_0(x) &= \begin{cases} \frac{1}{2}b_c\left(1 + \cos\frac{\pi x}{2}\right) & \text{for } -2 \le x \le 2 \\ 0 & \text{otherwise} \end{cases} \\ z_0(x) &= \frac{b_c}{(1+x^2)^{3/2}} \end{aligned}$$

Adaptive Time Step

- before each time step the maximum "pseudo-eigenvalue"

$$\lambda_{max} = \max_{i=1,2,x\in(A_x,B_x)} (u_i(x) \pm \sqrt{h_i(x)})$$

- current time step is then given by

$$\Delta t = \frac{C_{max}}{\lambda_{max}} \Delta x,$$

- C_{max} is the maximum Courant-Friedrichs-Levy (CFL) number which is used as an input parameter

One-layer Shallow Water

- comparision with analytical estimates
- 2000 points on the interval $x \in (-40, 40)$ with $C_{max} = 0.7$ for LF and LW time steps and $C_{max} = 2$ for the CN time steps, at $t = 20$

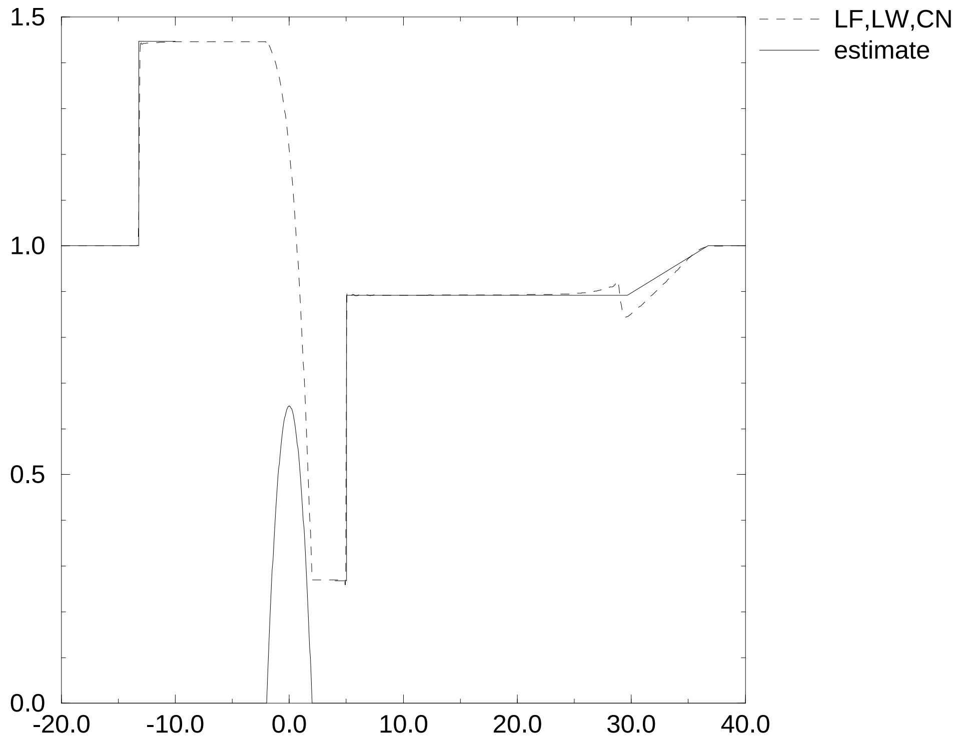

Figure 2: Comparision of heights of LFAD2, LWLF2 and CNLF1F8 methods (gave same results) with analytical estimates for the problem with $b_c = 0.65, u_0 = 0.7$.

One and Two-layer Shallow Water

- good agreement for same densities $R = \rho_2/\rho_1 = 1$
- 1000 points on the interval $x \in (-20, 20)$, the CFL number C_{max} was 0.7 for LF and LW time steps and 2 for CN time steps

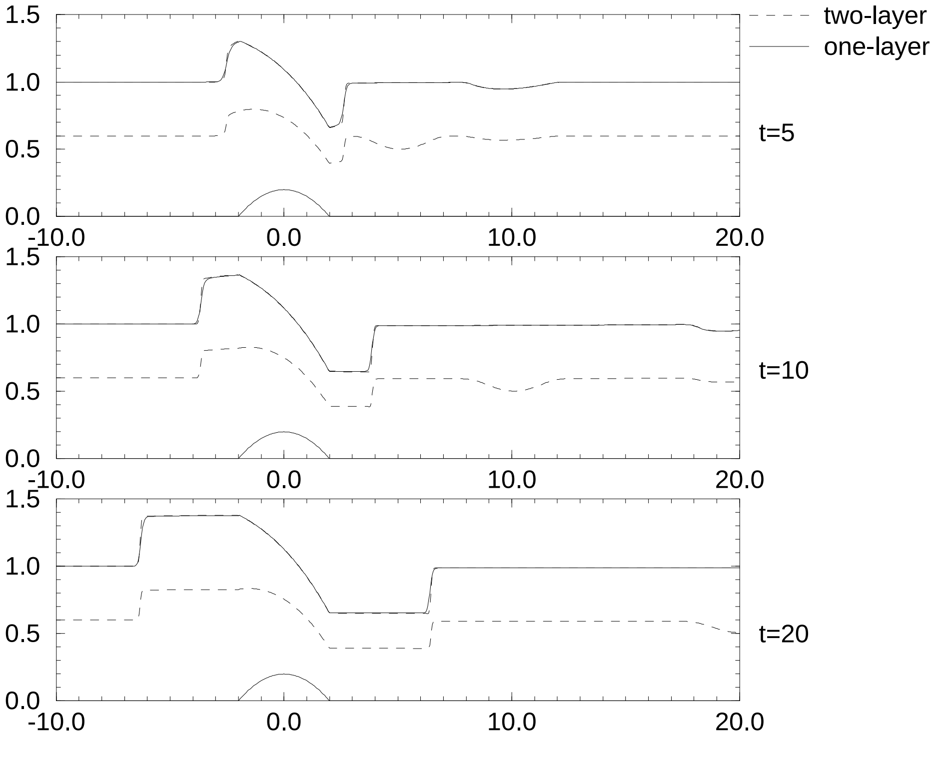

Figure 3: Comparision heights of two-layer (LFAD2, LWLF2, CNLF1F8 methods) and one-layer shallow water models for the problem with $b_c = 0.2, u_0 = 1, R = 1$ at times $t = 5, 10, 20$.

One and Two-layer Shallow Water

- disagreement for same densities $R = \rho_2/\rho_1 = 1$
- for other formulation including the equation for total momentum and difference of velocities instead of two momentum equations we have obtained a good agreement

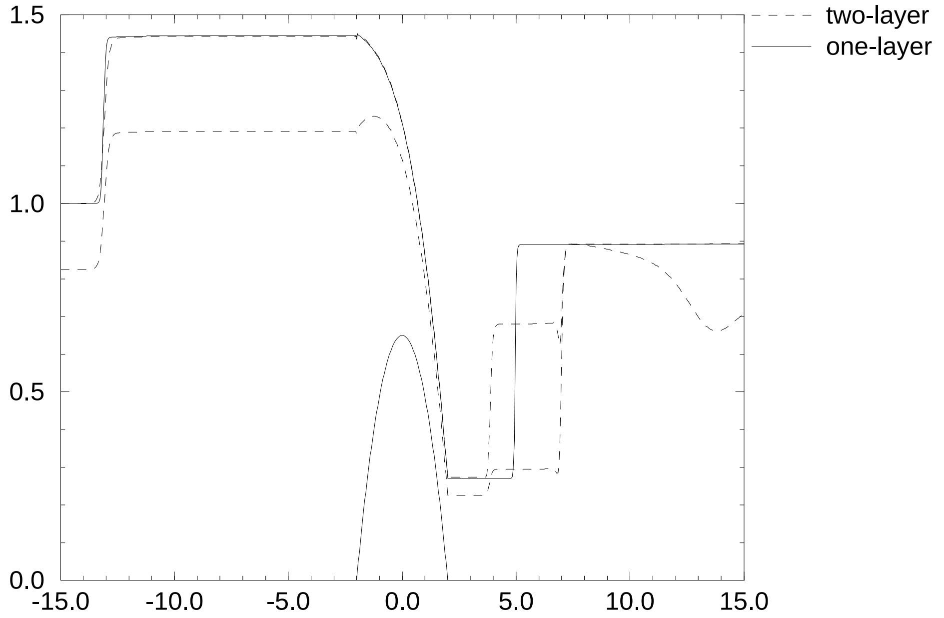

Figure 4: Comparision heights of two-layer and one-layer shallow water models for the problem with $b_c = 0.65, u_0 = 0.7$ at time $t = 20$ showing the disagreement of the two-layer model with the one-layer model, 1000 points on the interval $x \in (-20, 20)$ with C_{max} 0.35 for LFAD2 method, 0.2 for LWLF2 method and 0.5 for CNLF1F8 method with 0.2 for LF substeps

Two-layer Shallow Water

- different densities $R = \rho_2/\rho_1 = 0.8$
- 1000 points on the interval $x \in (-20, 20)$, with C_{max} 0.7 for LF and LW time steps and 2 for CN time steps

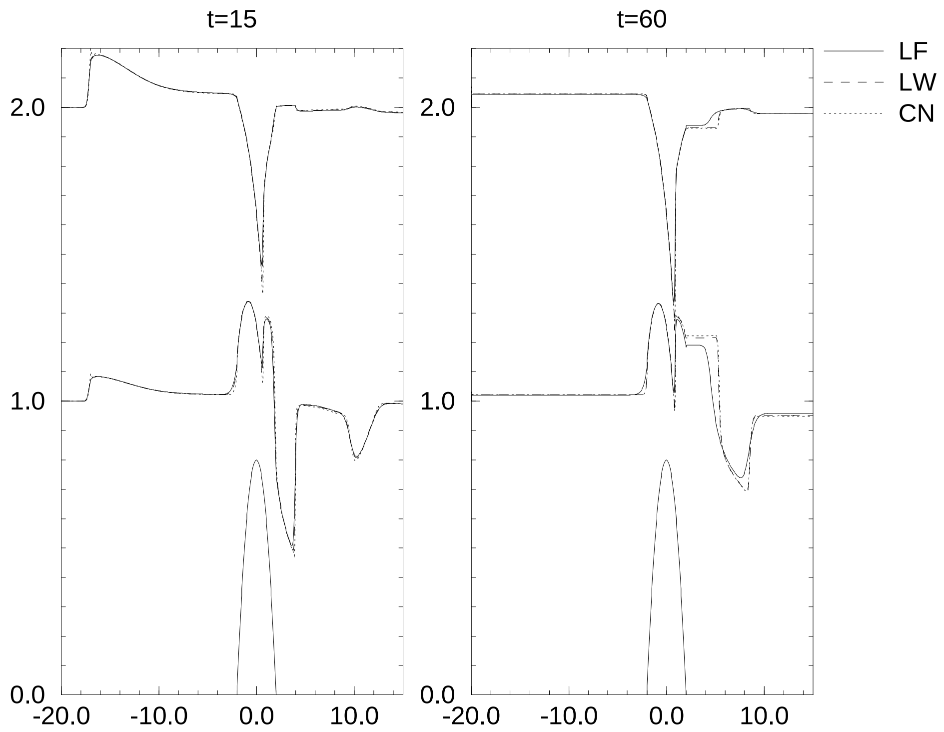

Figure 5: Comparision heights for two-layer shallow water model of LFAD2, LWLF4 and CNLF1F8 methods for the problem with $b_c = 0.8, u_0 = 0.4, R = 0.8$ at times $t = 15, 60$.

Two-layer Shallow Water

- different densities $R = \rho_2/\rho_1 = 0.8$
- 2000 points on the interval $x \in (-20, 20)$, and we have used C_{max} 0.35 for LF method, 0.2 for LW method and 0.5 for CN method with 0.2 for LF time substeps

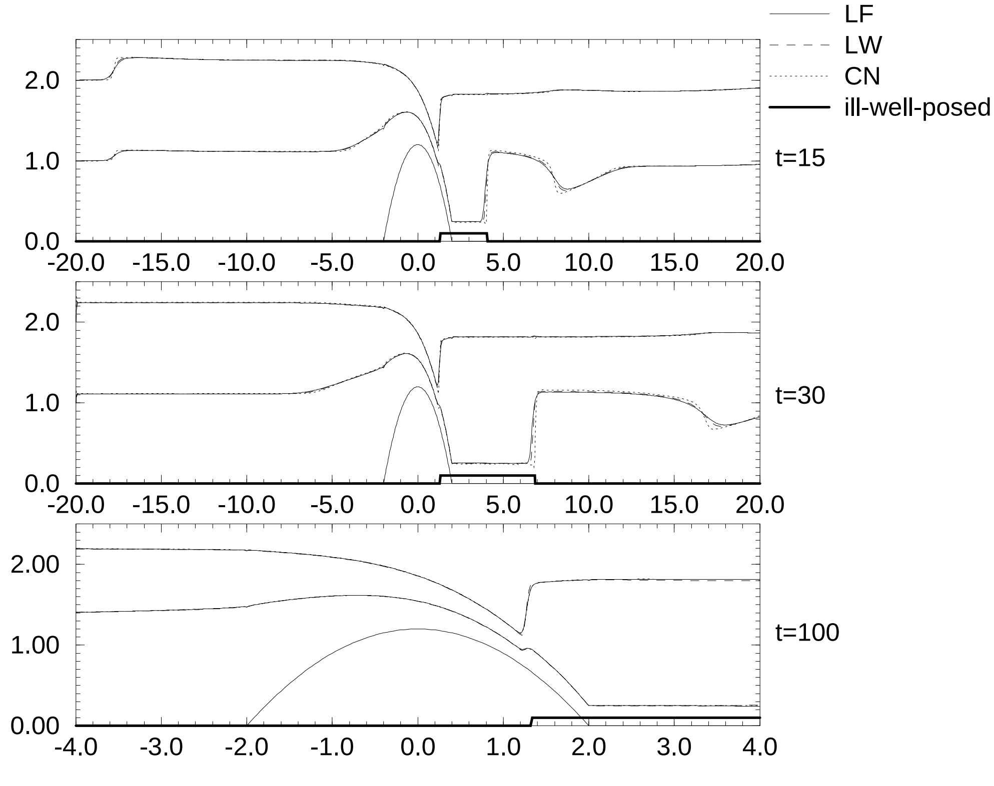

Figure 6: Comparision heights for two-layer shallow water model of LFAD2, LWLF4 and CNLF1F8 methods for the problem with $b_c = 1.2, u_0 = 0.4, R = 0.8$ at times $t = 15, 30, 100$. The bold line distinquishes the well-posed (value 0) and ill-posed (value 0.1) regions.

One-layer Green-Naghdi

- comparision of our BEN Green-Naghdi code with semi-Lagrangian (SL) code (Nadiga, Margolin, Smolarkiewicz, 1996)
- case $u_0 = 0.7, b_c = 0.65$
- 2048 points on the interval $x \in (-40, 40)$ and $C_{max} = 0.8$

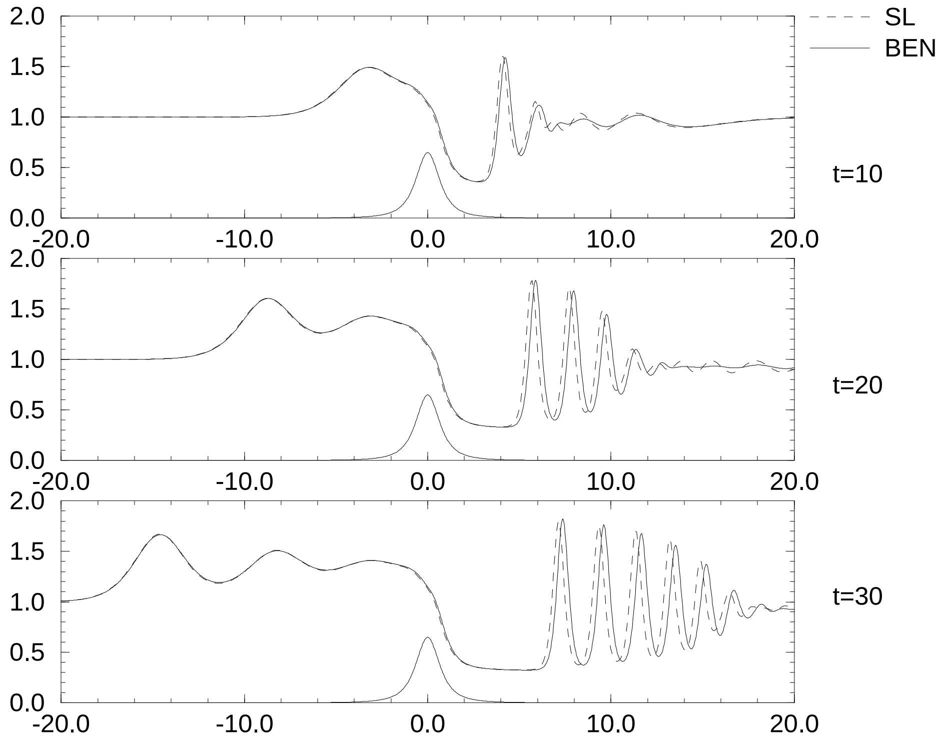

Figure 7: Comparision of heights z_1 for one-layer GN semi-Lagrangian code with our one-layer GN code

One-layer Green-Naghdi and Boundary Integral Formulation

- comparision with boundary integral formulation (Belward, Forbes, 1995), steady state solution
- case $z_1^0 = 1, u_1^0 = 1/\sqrt{2}, b_c = 0.056$

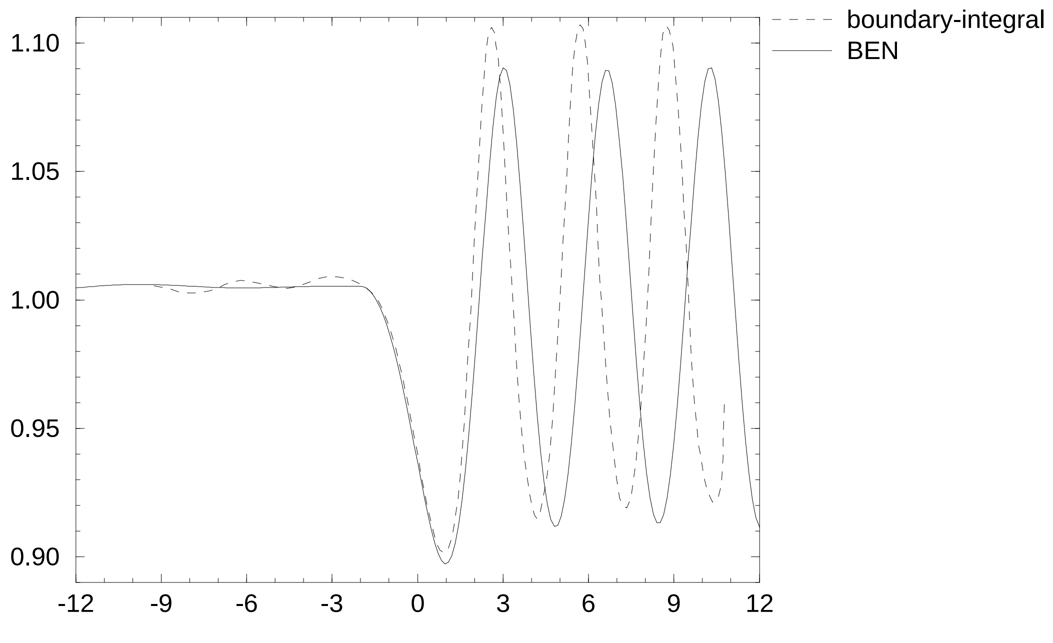

Figure 8: Comparision of heights for steady state solution obtained by the boundary integral formulation and our one-layer GN code, 2000 points on the interval $x \in (-80, 80)$ with $C_{max} = 1.6$.

One- and Two-layer Green-Naghdi

- case $z_1^0 = 0.5, z_2^0 = 1, u_1^0 = u_2^0 = 1, R = 1, b_c = 0.2$

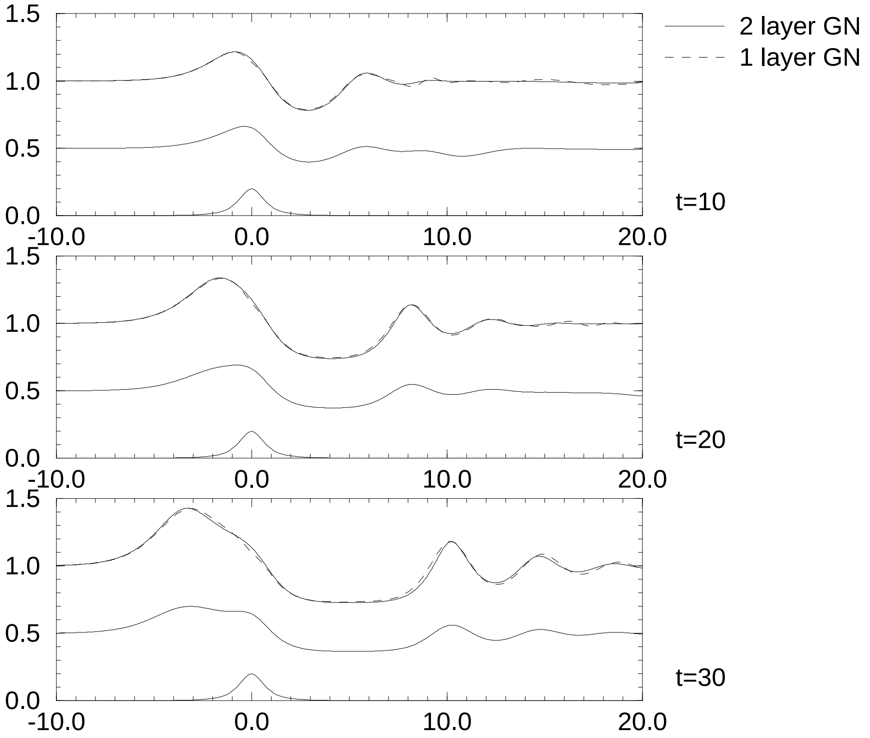

Figure 9: Comparision of heights for GN one- and two-layer codes

Comparision with Euler

- one- and two-layer Green-Naghdi codes compared with Euler code (Nadiga, Margolin, Smolarkiewicz, 1996)
- case $z_1^0 = 0.5, z_2^0 = 1, u_1^0 = u_2^0 = 1, R = 1, b_c = 0.2$

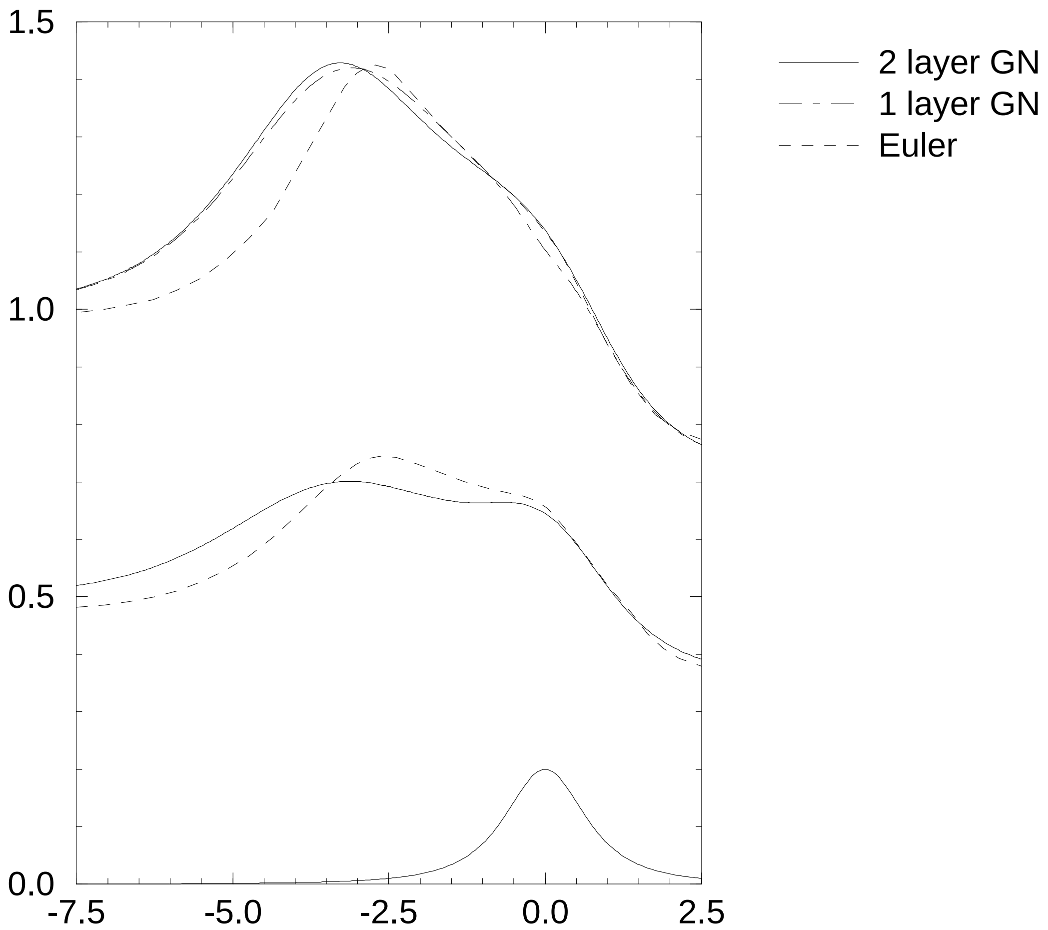

Figure 10: Comparision of heights z_1, z_2 of the two-layer Green-Naghdi (GN) code with an Euler code for layers (0,0.5) and (0.5,1)

Convergence Test

- case $z_1^0 = 0.9, z_2^0 = 1, u_1^0 = u_2^0 = 1, R = 1, b_c = 0.2$
- 4 sucessive grid refinements with 500, 1000, 2000, 4000 points on the interval $x \in (-40, 40)$ with the CFL number $C_{max} = 2.5$
- instability for the fine grids around $t = 40, x = -1$

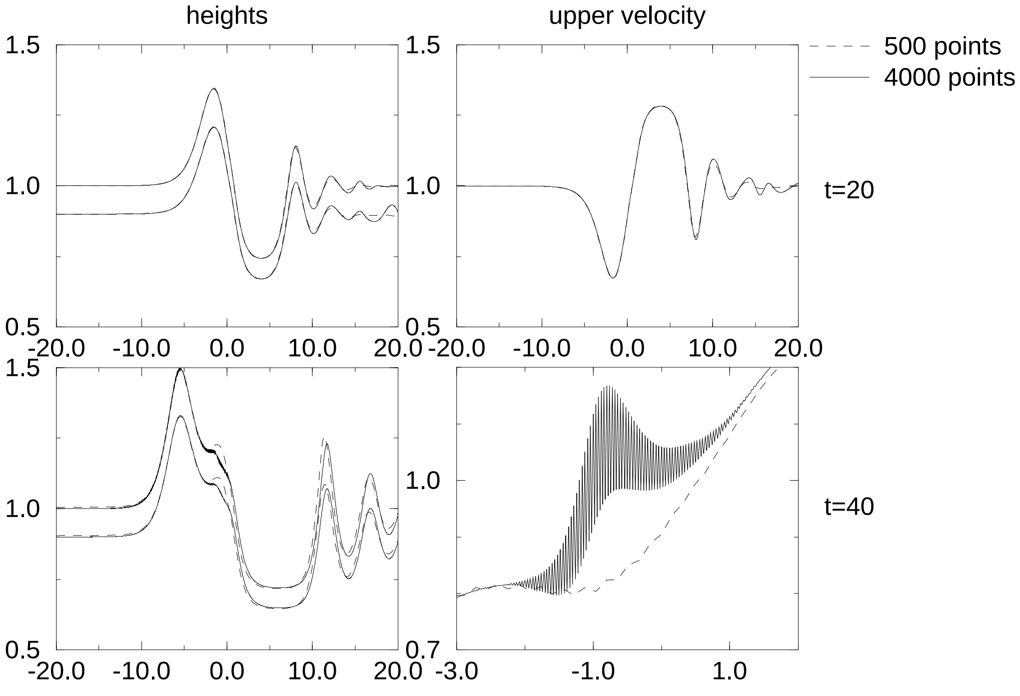

Figure 11: Convergence test for two-layer Green-Naghdi with interface at $z_1^0 = 0.9$ at times $t = 20, 40$, heights z_1, z_2 and velocity of the upper layer u_2.

Conclusion

- two-layer models can be extended to n-layer models
- can be done also for 2D vertically averaged models derived from full 3D Euler equations
- shallow-water and Green-Naghdi models
- computation with these models is possible although there are difficulties connected with their ill-posedness and other properties
- results qualitatively agree with results obtained by more detailed models
- composite difference schemes