Invariant Variational Principles and Associated Numerical Schemes for Regularization of Ill-Posed Problems

R. C. Venkatesan

Systems Research Corporation
Aundh, Pune 411007, INDIA

ABSTRACT

Rigorous regularization theories for providing approximate solutions to ill-posed problems have been developed over the past few years [1]. The underlying concept of regularization techniques is to restrict the space of acceptable solutions by choosing the function that minimizes an appropriate functional. The regularization of the ill-posed problem of finding "z" from observational data "y" such that $\boldsymbol{A}z=y$ requires the choice of norms and a stabilization functional (in the sense of Tikhonov).

Conventional techniques employed in solution of Tikhonov regularization problems employ either an optimization approach, or minimization of a cost functional (usually defined in a Hilbert space) via classical variational methods .

The present work first presents a general framework using Nöether's theorem [2] for solving ill-posed problems via minimization of a given variational of the form:

$$H[f] = \sum_{i=1}^{N} \left[y - f(\mathbf{x}_i) \right]^2 + \lambda \left\| \mathbf{P} f(\mathbf{x}_i) \right\|^2,$$

where $f(\mathbf{x_i})$ is the function to be reconstructed/approximated from the observational data, λ is the regularization parameter and **P** is a differential operator. Nöther's theorem provides an algorithm to construct conservation laws for systems described by a Lagrangean function, for any admitted infinitesimal transformation. By definition, conservation laws are expressions of the form $\nabla \bullet \Phi = 0$. The notion of conservation laws is in a sense dual to that of a symmetry admitted by a given set of differential equations. Nöther's theorem establishes the relationship between conservation laws and symmetries for an Euler-Lagrange equation. Methodologies for obtaining optimal values of the regularization parameter and the Tikhonov stabilizers on the basis of infinitesimal transformations possessed by the conservation laws, are described [3].

Next, the formulation of finite difference approximations for the conservation laws which possess groups of infinitesimal transformations defined in mesh space which are isomorphic to those possessed by the original variational for the ill-posed problem, are described [4,5]. Generic guidelines for the numerical implementation of these schemes using Computer Algebraic Systems are suggested. Exemplary numerical examples for cases in computer vision (edge detection) and statistical learning are provided.

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