Group Invariant Finite-Difference Schemes for the Advection Equation

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Abstract

The present work derives a set of invariant finite-difference approximations to the scalar advection equation:

$$u_{,t} + \alpha u_{,x} = 0,$$

where $u_{,t} = \frac{\partial u}{\partial t}$, $u_{,x} = \frac{\partial u}{\partial x}$, and, α is a constant. The advection equation is a prototypical hyperbolic equation in fluid dynamics, whose use in CFD rivals even that of the diffusion equation. It also serves as a limiting form for a number of important equations, such as, the equations of radiative transfer, and primarily describes how a conserved quantity such as potential temperature or momentum is carried along with a flow of air or water.

The preservation of invariance is known to play an important role in choosing the type of discretization scheme to be employed, during the numerical solution of systems of differential equations [1]. More specifically, preservation of invariance by a given numerical scheme ensures that physical phenomena described by the continuous model being approximated are accurately modeled.

Invariant finite difference schemes are numerical approximations to systems of differential equations, which possess groups that are isomorphic to those of the original system of differential equations. By definition, in Lie group theory, given two groups G and H a homomorphism is a smooth map $\phi: G \to H$ exists such that:

$$\phi(g \cdot \tilde{g}) = \phi(g) \cdot \phi(\tilde{g}), g, \tilde{g} \in G.$$

If ϕ has a smooth inverse, then there exists an isomorphism between groups G and H.

A number of approaches which yield invariant finite difference schemes have been investigated into. The methodology followed in the present work

closely resembles the theory propounded by Dorodnitsyn [2]. The primary objective of such an approach is to discretize a given differential equation in a manner such that the resulting finite-difference equation inherits all Lie point symmetries possessed by the original differential equation.

The distinct feature of such a formulation lies in the fact that the group transformations act on the variables and the derivative terms as well as on the lattice which defines the grid. This entails the fact that lattices are prescribed for specific difference equations, so as to render the groups defined in grid space isomorphic to the groups possessed by the differential equations. An implied consequence is that invariant finite difference approximations to differential equations are not unique.

The present work describes using Computer Algebraic Systems to solve invariant finite-difference Godunov and Total Variation Diminishing Schemes [3] for the advection equation for both uniform and nonuniform grids. The derived invariant finite difference schemes are compared with some of their non-invariant ("traditional") counterparts. Finally, stability and error analyses are performed, which conclusively demonstrate the superior numerical characteristics of invariant schemes vis-a-vis their non-invariant counterparts. Numerical examples for exemplary cases are provided.

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