Derivation of Alfven wave from non-ideal MHD

$$\eta = \frac{1}{\sigma} \qquad \eta \vec{j} = \vec{E} + \vec{v} \times \vec{B}$$

$$\frac{\partial \vec{B}}{\partial t} = -\mathrm{rot}\,\vec{E} = \mathrm{curl}(\vec{v} \times \vec{B}) - \mathrm{curl}(\eta \vec{j})$$

$$\mathrm{curl}\,\vec{B} = \mu_0 \vec{j}$$

$$\rho_0\left(\frac{\partial \vec{v}}{\partial t} + (\vec{v}\nabla)\vec{v}\right) = -\nabla p + \vec{j} \times \vec{B}$$

$$\frac{\partial \rho}{\partial t} + \mathrm{div}(\rho v) = 0$$

$$\vec{B} = B_0 \hat{z} + B_1 \hat{x} \qquad \vec{k} = (0, 0, \vec{k})$$

$$\vec{v} = \vec{0} + v_1 \hat{x} \qquad \vec{j} = \vec{0} + j_1 \hat{y}$$

$$\vec{v} \perp z \Rightarrow \rho = \rho_0 \qquad \text{isothermal} \rightarrow \nabla p = 0$$

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linearization

$$\frac{\partial \vec{B}_1}{\partial t} = \mathrm{curl}(\vec{v}_1 \times B_0) - \eta\,\mathrm{curl}\,j_1$$

$$\mathrm{curl}\,\vec{B}_1 = \mu_0 \vec{j_1}$$

$$\rho_0 \frac{\partial \vec{v}_1}{\partial t} = \vec{j}_1 \times \vec{B}_0$$

$\Rightarrow$

$$\frac{\partial B_1}{\partial t} = B_0 \frac{\partial v_1}{\partial z} + \eta \frac{\partial j_1}{\partial z}$$

$$\frac{\partial B_1}{\partial z} = \mu_0 j$$

$$\rho_0 \frac{\partial v_1}{\partial t} = j_1 B_0$$

Solution $e^{ikz - i\omega t}$

$$-i\omega B_1 = ikB_0 v_1 + ik\eta j_1$$

$$ikB_1 = \mu_0 j_1$$

$$-i\omega \rho_0 v_1 = j_1 B_0$$

$$-\omega B_1 = +k\left[\frac{i}{\omega\rho_0}\right] j_1 B_0^2 + k\eta j_1$$

$$\omega B_1 = -k\left[\frac{iB_0^2}{\omega\rho_0} + \eta\right]\frac{ikB_1}{\mu_0}$$

dispersion equation

$$\omega^2 = k^2\left[\frac{B_0^2}{\mu_0\rho_0} - i\omega\eta\right]$$

$\eta = 0 \Rightarrow \omega^2 = k^2 v_A^2$ Alfven wave

$\eta > 0$ $Re(\omega) > 0$ $Im(\omega) < 0$ damping

for small resistivity $k\eta \ll v_A$

$$\omega = |k| v_A - \frac{i\eta k^2}{2}$$

Solution $\simeq A e^{ikz} e^{-i|k|v_A t} e^{-\frac{\eta k^2}{2}t}$

for real ω $k = \pm \frac{\omega}{\sqrt{v_A^2 + i\omega\eta}}$ $\simeq \pm \frac{\omega}{v_A\sqrt{1+\frac{\omega^2\eta^2}{v_A^4}}}\left(1 + \frac{i}{2}\frac{\omega\eta}{v_A^2}\right)$