Waves – Repetition – Seminar

Wave $a = A(\vec{r},t)\, e^{i(\vec{k}\vec{r} - \omega t)}$ $(\vec{a}, \vec{A})$

Amplitude slowly varying $\cong$ constant

Phase $\varphi = \vec{k}\vec{r} - \omega t$ $\varphi = \mathrm{const}$ wavefront

$$\vec{k} = \frac{\partial \varphi}{\partial \vec{r}} = \nabla \varphi$$ $\vec{k}$ normal to wavefront

$$\vec{r} = \vec{r}_0 + \vec{v}_\varphi t$$

φ = const.

$$\varphi = \vec{k}\vec{r}_0 + (\vec{k}\vec{v}_\varphi - \omega)t \Rightarrow \vec{v}_\varphi = \frac{\omega}{k}\cdot\frac{\vec{k}}{k}$$

$$k = |\vec{k}|$$

$$|\vec{v}_\varphi| = \frac{\omega}{k}$$

Group velocity – how a signal propagates – plane wave brings no information

$$a = \int A_{\vec{k}}\, e^{i[\vec{k}\vec{r} - \omega(\vec{k})t]}\, d\vec{k}$$

Wave packet

Lets assume a narrow spectrum around ω_0, $\vec{k}_0$, where $\omega_0 = \omega(\vec{k}_0)$

$$a = e^{i(\vec{k}_0\vec{r} - \omega_0 t)} \underbrace{\int A_{\vec{k}}\, e^{i(\vec{k}-\vec{k}_0)\vec{r} - i(\omega(\vec{k}) - \omega_0)t}\, d(\vec{k}-\vec{k}_0)}_{\text{envelope}}$$

$$\omega(\vec{k}) = \omega_0 + \left.\frac{\partial \omega}{\partial \vec{k}}\right|_{\vec{k}_0} (\vec{k} - \vec{k}_0) +$$

$$a = e^{i(\vec{k}_0\vec{r} - \omega_0 t)} \int A_{\vec{k}}\, e^{i(\vec{k}-\vec{k}_0)\left(\vec{r} - \frac{\partial\omega}{\partial\vec{k}} t\right)} d(\vec{k}-\vec{k}_0)$$

Envelope moves with velocity $\vec{v}_g = \left.\frac{\partial\omega}{\partial\vec{k}}\right|_{\omega=\omega_0}$ (need not be parallel to v_φ) and group velocity is less than or equal to the speed of light *c*.

In non-dispersive media, v_φ does not depend on $\vec{k} \Rightarrow \vec{v}_g = \vec{v}_\varphi$

Group velocity and energy transport

For electromagnetic waves, plasma is a medium with temporal dispersion

$\varepsilon_r = \varepsilon_r(\omega)$ $\quad k^2 = \frac{\omega^2}{c^2}\varepsilon_r \quad$ let $\vec{k} = (k,0,0)$

$$\vec{E} = \vec{E}_0\, e^{i(kx-\omega t)}$$

$$H = \frac{kE_0}{\mu_0\omega}\, e^{i(kx-\omega t)} \qquad E = E_y \qquad H = H_z$$

Poynting vector (energy flux density)

$$\vec{S} = \frac{1}{2}\left(\vec{E}\times\vec{H}^*\right) = \hat{x}\,\frac{1}{2}\,\frac{kE_0^2}{\mu_0\omega} = \frac{1}{2}\,\frac{\sqrt{\varepsilon_r}\,E_0^2}{\mu_0 c}$$

$$= \frac{1}{2}\sqrt{\frac{\varepsilon_0\varepsilon_r}{\mu_0}}\,E_0^2 = \frac{1}{2}c\sqrt{\varepsilon_r}\,\varepsilon_0 E_0^2$$

Phase and group velocity

$$v_\varphi = \frac{\omega}{k} = \frac{c}{\sqrt{\varepsilon_r}} > c$$

$$v_g = \frac{1}{\frac{dk}{d\omega}} = \frac{c}{\frac{d}{d\omega}\left(\omega\sqrt{\varepsilon_r}\right)} = \frac{c\sqrt{\varepsilon_r}}{\varepsilon_r + \frac{1}{2}\omega\frac{d\varepsilon_r}{d\omega}} = \frac{c\sqrt{\varepsilon_r}}{1 - \frac{\omega_p^2}{\omega^2} + \frac{\omega_p^2}{\omega^2}} = c\sqrt{\varepsilon_r} < c$$

Speed of energy transport

$$\vec{S} = \vec{v}_W \cdot W$$

$$W = \frac{1}{4}\frac{d}{d\omega}(\omega\varepsilon)|E_0|^2 + \frac{1}{4}\mu_0|H_0|^2 =$$

$$= \frac{1}{4}\varepsilon_0\varepsilon_r|E_0|^2 + \frac{1}{4}\frac{d\varepsilon_r}{d\omega}\omega\varepsilon_0|E_0|^2 +$$

$$+ \underbrace{\frac{1}{4}\mu_0\frac{\omega^2}{c^2}\frac{\varepsilon_r}{\mu_0^2\omega^2}|E_0|^2}_{\frac{1}{4}\varepsilon_0\varepsilon_r|E_0|^2} = \frac{1}{2}\varepsilon_0\varepsilon_r|E_0|^2 + \frac{1}{4}\frac{d\varepsilon_r}{d\omega}\omega\varepsilon_0|E_0|^2$$

$$v_W = \frac{S}{W} = \frac{\frac{1}{2}c\sqrt{\varepsilon_r}\,\varepsilon_0|E_0|^2}{\frac{1}{2}\left(\varepsilon_0\varepsilon_r + \frac{1}{2}\varepsilon_0\omega\frac{d\varepsilon_r}{d\omega}\right)|E_0|^2} =$$

$$= \frac{c\sqrt{\varepsilon_r}}{\varepsilon_r + \frac{1}{2}\omega\frac{d\varepsilon_r}{d\omega}} = v_g$$