Degenerate electron gas

Fermions = maximum 1 fermion in any state

Phase volume on 1 fermion

$$\Delta p \cdot \Delta V = \frac{h^3}{n_{sp}}$$

n_{sp} – number of spin orientations

electron spin +1/2, -1/2 $\Rightarrow$ $n_{sp} = 2$

for T= 0 - all states with energy $\mathcal{E}$ < Fermi energy $\mathcal{E}_F$ are occupied and all higher states are empty

$$\mathcal{E} = \frac{p^2}{2m} \qquad p_F = \sqrt{2m\mathcal{E}_F}$$

N identical fermions in volume V

$$N = \frac{V n_{sp}}{h^3}\int_0^{p_F} 4\pi p^2 dp = V\frac{4\pi n_{sp} p_F^3}{3h^3} = V\frac{4\pi n_{sp}(2m\mathcal{E}_F)^{3/2}}{3h^3}$$

$$\Rightarrow \mathcal{E}_F = \frac{h^2}{2m}\left(\frac{3n}{4\pi n_{sp}}\right)^{2/3}$$

electrons $\mathcal{E}_F = \frac{h^2}{2m_e}\left(\frac{3n_e}{8\pi}\right)^{2/3}$

Electrons $\mathcal{E}_F \cong 7.9\ eV$ for $n_e = 10^{23}\,cm^{-3} = 10^{29}\,m^{-3}$

Protons $\mathcal{E}_F \cong 50\ K$ for $n_i = 10^{23}\,cm^{-3} = 10^{29}\,m^{-3}$

Electron internal energy for T = 0

$$U_e = V\cdot\frac{8\pi}{h^3}\int_0^{p_F}\frac{p^2}{2m_e}p^2dp = \frac{8\pi V}{h^3}\frac{1}{2m_e}\frac{p_F^5}{5} = \frac{3}{5}n_e V\mathcal{E}_F = \frac{3}{5}N\mathcal{E}_F$$

Electron pressure at T = 0 (for T = 0 entropy S = 0)

$$p_e = -\left(\frac{\partial U_e}{\partial V}\right)_{S,N} = -\frac{\partial}{\partial V}\left[\frac{3}{5}N\frac{h^2}{2m_e}\left(\frac{3N}{8\pi V}\right)^{2/3}\right] = \frac{2n_e}{5}\left[\frac{h^2}{2m_e}\left(\frac{3n_e}{8\pi}\right)^{2/3}\right]$$

$$p_e = \frac{2}{5}n_e\mathcal{E}_F \Rightarrow U_e = \frac{3}{2}p_e V$$

Electrons – Fermi-Dirac distribution

$$n_e = \frac{8\pi}{h^3}\int_0^\infty \frac{p^2\,dp}{e^{\frac{\varepsilon_e-\mu}{k_B T_e}}+1} \qquad \varepsilon_e = \frac{p^2}{2m_e}$$

Chemical potential μ - energy necessary for adding 1 particle at S = const. and V = const.

For $T = 0$ chemical potential $\mu = \varepsilon_F$ degeneracy parameter θ_D

$$\frac{\mu}{k_B T_e} = \frac{1}{\theta_D} \qquad \left(\theta_D = \frac{k_B T_e}{\varepsilon_F}\right)$$

Opposite extreme $T_e >> \varepsilon_F$ ideal gas – 1 in the denominator can be neglected

$$n_e = \frac{8\pi}{h^3}\int_0^\infty p^2\,dp\, e^{\frac{\mu-\varepsilon_e}{k_B T_e}}$$

$$e^{-\frac{\mu}{k_B T_e}} = \frac{8\pi}{n_e h^3}\int_0^\infty p^2 e^{-\frac{p^2}{2m_e k_B T_e}}\,dp = \frac{2(2\pi m_e k_B T_e)^{3/2}}{n_e h^3} = \frac{3\sqrt{\pi}\,\theta_D^{3/2}}{4}$$

$$\frac{\mu}{k_B T_e} = \ln\left(\frac{4}{3\sqrt{\pi}}\right) - \frac{3}{2}\ln\theta_D \qquad \text{при } \theta_D = \theta_{crit} = 0.827,\ \mu = 0$$

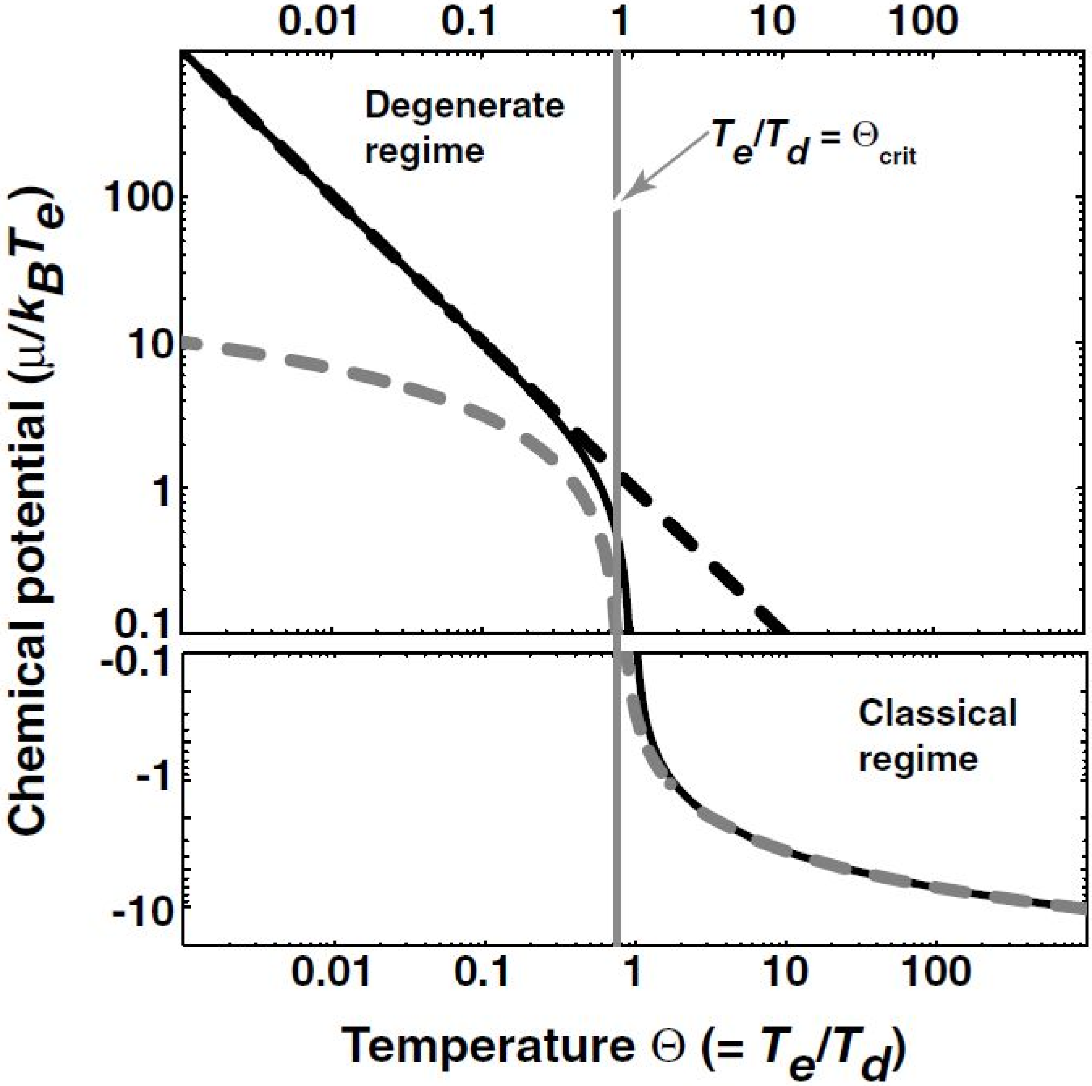
$$\mu < 0 \text{ при } \theta_D > \theta_{crit}$$

Ideal gas - particle with $\varepsilon = 0$ can be added, but S is increased $\Rightarrow$ energy must be removed

Fit for any degeneracy (book by Ichimaru)

$$\frac{\mu}{k_B T_e} = -\frac{3}{2}\ln\theta_D + \ln\left(\frac{4}{3\sqrt{\pi}}\right) + \frac{0.2505\,\theta_D^{-1.858} + 0.072\,\theta_D^{-\frac{1.858}{2}}}{1 + 0.2505\,\theta_D^{-0.858}}$$

Figure – full line – accurate value, dotted – asymptotic formulae derived above


0.01
0.1
1
10
100
Degenerate
regime
$T_e/T_d = \Theta_{crit}$
Chemical potential ($\mu/k_B T_e$)
100
10
1
0.1
-0.1
-1
-10
Classical
regime
0.01
0.1
1
10
100
Temperature Θ (= T_e/T_d)