Examples of Fokker-Planck equation solving

A) Electron-ion relaxation

It is process of temperature equalization between electrons and ions. In first approximation one can assume homogeneous neutral plasma, and thus distribution functions vary in time only as a result of collisions

$$\frac{\partial f_a}{\partial t} = \frac{\partial}{\partial p_{ak}} \sum_b \frac{q_a^2 q_b^2}{8\pi\varepsilon_0^2} \ln\Lambda_{ab} \int \mathrm{d}\vec{p}_b \, \frac{\mathrm{v}_{ab}^2 \, \delta_{kl} - \mathrm{v}_{abk} \, \mathrm{v}_{abl}}{\mathrm{v}_{ab}^3} \left(\frac{\partial f_a}{\partial p_{al}} f_b - f_a \frac{\partial f_b}{\partial p_{bl}} \right)$$

As momentum relaxation proceeds faster than temperature balancing, we can assume that electrons and ions have Maxwell's distributions with diff. tempers

$$f_a = \frac{n_a}{\left(2\pi m_a k_B T_a\right)^{3/2}} \exp\left(-\frac{p_a^2}{2 m_a k_B T_a} \right)$$

For Maxwell's distribution

$$\frac{\partial f_a}{\partial p_{al}} f_b - f_a \frac{\partial f_b}{\partial p_{bl}} = f_a f_b \left(\frac{\mathrm{v}_{bl}}{k_B T_b} - \frac{\mathrm{v}_{al}}{k_B T_a} \right)$$

and for temperatures $T_a = T_b$ this expression is the component of vector parallel with mutual velocity of particles and collision integral kernel is orthogonal to such vector, and thus collisions of particles with the same temperature produce no contribution.

We shall calculate the second moment of distribution function

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{3}{2}n_a k_B T_a\right) = -\sum_b \frac{q_a^2 q_b^2}{8\pi\varepsilon_0^2 m_a} \ln\Lambda_{ab} \iint \mathrm{d}\vec{p}_a \, \mathrm{d}\vec{p}_b \mathrm{v}_{ak} \, \frac{\mathrm{v}_{ab}^2 \, \delta_{kl} - \mathrm{v}_{abk} \, \mathrm{v}_{abl}}{\mathrm{v}_{ab}^3} \times$$

$$\times f_a f_b \left(\frac{\mathrm{v}_{bl}}{k_B T_b} - \frac{\mathrm{v}_{bl}}{k_B T_a} - \frac{\mathrm{v}_{abl}}{k_B T_a} \right)$$

Last term in brackets is orthogonal, thus it does not contribute and thus

$$\frac{\mathrm{d}T_a}{\mathrm{d}t} = -\sum_b \nu_{Tab} \left(T_a - T_b\right)$$

where collision frequency for energy exchange is

$$\nu_{Tab} = \frac{q_a^2 q_b^2}{12\pi\varepsilon_0^2 k_B T_a k_B T_b} \ln\Lambda_{ab} \frac{n_b}{\left(2\pi k_B\right)^3 \left(m_a T_a m_b T_b\right)^{3/2}} \iint \mathrm{d}\vec{p}_a \, \mathrm{d}\vec{p}_b \mathrm{v}_{ak} \mathrm{v}_{bl} \times$$

$$\times \frac{\mathrm{v}_{ab}^2 \, \delta_{kl} - \mathrm{v}_{abk} \, \mathrm{v}_{abl}}{\mathrm{v}_{ab}^3} \ \exp\left(-\frac{p_a^2}{2 m_a k_B T_a} - \frac{p_b^2}{2 m_b k_B T_b} \right)$$

Let index a denotes electrons, index b ions. Quadratic term in expansion via small ion velocity is the lowest that provides nonzero contribution.
Then the integral is expressed as follows

$$\iint \mathrm{d}\vec{p}_e\, \mathrm{d}\vec{p}_i \mathrm{v}_{ik} \mathrm{v}_{il}\, \frac{\mathrm{v}_e^2\, \delta_{kl} - \mathrm{v}_{ek}\, \mathrm{v}_{el}}{\mathrm{v}_e^3}\, \exp\left(-\frac{p_e^2}{2m_e k_B T_e} - \frac{p_i^2}{2m_i k_B T_i} \right) = \frac{2}{3} \int \frac{\mathrm{d}\,\vec{p}_e}{\mathrm{v}_e} \times$$

$$\times \exp\left(-\frac{p_e^2}{2m_e k_B T_e} \right) \int \mathrm{d}\vec{p}_i\, \mathrm{v}_i^2 \exp\left(-\frac{p_i^2}{2m_i k_B T_i} \right) = 8\,\pi^{5/2} m_e^2 k_B T_e \left(2m_i k_B T_i \right)^{5/2}$$

and after substitution the collision frequency reads

$$\nu_{Tei} = \frac{2m_e}{m_i} \frac{4}{3} \frac{\sqrt{2\pi}\, q_e^2\, q_i^2\, n_i}{\sqrt{m_e}\, \left(4\pi\varepsilon_0\right)^2 \left(k_B T_e\right)^{3/2}} \ln \Lambda_{ei} = \frac{2m_e}{m_i}\, \nu_{ei}$$

where ν_{ei} is effective collision frequency for momentum transfer. Here, we could not assume nonmoving ions, as there would be no energy transfer.

B) Symmetrizing of electron distribution function

Often, it happens that temperature electron $T_{||}$ along magnetic field differs from temperature $T_{\perp}$ in direction perpendicular to magnetic field. Thus, Maxwell’s distribution with different longitudinal and transverse temperatures is assumed

$$f_e\left(\vec{p}, t\right) = \frac{n_e}{2\pi m_e k_B T_{\perp}\left(t\right) \sqrt{2\pi m_e k_B T_{||}\left(t\right)}} \exp\left(-\frac{p_x^2 + p_y^2}{2m_e k_B T_{\perp}\left(t\right)} - \frac{p_z^2}{2m_e k_B T_{||}\left(t\right)} \right)$$

We omit for simplicity collisions between electrons, which is possible if $q_i \gg |q_e|$. We assume nonrelativistic temperatures. If ion temperature is not substantially higher than electron temperature, we can omit ion velocities in comparison with electron ones and electrons change the motion direction during scattering on non-movable heavy scattering centers. This assumption simplifies collision integral significantly, the term with the derivative of ion distribution disappears after integration and

$$\frac{\partial f_e}{\partial t} = \frac{q_e^2 q_i^2 n_i m_e}{8\pi \varepsilon_0^2} \ln \Lambda_{ei} \frac{\partial}{\partial p_{ek}} \left(\frac{p_e^2 \delta_{kl} - p_{ek} p_{el}}{p_e^3} \frac{\partial f_e}{\partial p_{el}} \right) = \frac{q_e^2 q_i^2 n_i}{8\pi \varepsilon_0^2} \ln \Lambda_{ei} \times$$

$$\times \frac{\partial}{\partial p_{ek}} \left\{ f_e \frac{p_e^2 \delta_{kl} - p_{ek} p_{el}}{p_e^3} \left[-\frac{p_x \delta_{xl} + p_y \delta_{yl} + p_z \delta_{zl}}{k_B T_\perp} + p_z \delta_{zl} \left(\frac{1}{k_B T_\perp} - \frac{1}{k_B T_\parallel} \right) \right] \right\}$$

The first term in the square brackets disappears after summation (it corresponds to the symmetric Maxwell’s distribution) and thus

$$\frac{\partial f_e}{\partial t} = \frac{T_\parallel - T_\perp}{k_B T_\parallel T_\perp} \frac{q_e^2 q_i^2 n_i}{8\pi \varepsilon_0^2} \ln \Lambda_{ei} \frac{\partial}{\partial p_{ek}} \left(\frac{p_e^2 \delta_{kz} - p_{ek} p_{ez}}{p_e^3} p_{ez} f_e \right)$$

We multiply this kinetic equation by terms $\left(p_x^2+p_y^2\right)/2m_e n_e$ a $p_z^2/2m_e n_e$ and integrate over momentum space. We obtain equations for temperature evolution

where at small temperature difference $T_\perp \simeq T_\parallel \simeq T$, the collision frequency is

$$\nu_p \cong \frac{q_e^2 q_i^2 n_i}{8\pi\varepsilon_0^2 m_e \left(k_B T\right)^2} \ln\Lambda_{ei} \int \frac{\mathrm{d}\vec{p}}{\left(2\pi m_e k_B T\right)^{3/2}} \exp\left(-\frac{p^2}{2m_e k_B T}\right) p_z^2 \frac{3p_z^2-p^2}{p^3} = \frac{4}{5}\nu_{ei}$$

If one additionally includes mutual electron collisions, then the collision frequency reads, as follows

$$\nu_p = \frac{4}{5}\nu_{ei}\left(1+\left|\frac{q_e}{q_i}\right|\frac{1}{\sqrt{2}}\right)$$

$$\frac{\mathrm{d}}{\mathrm{d}t}T_\parallel = -\nu_p\left(T_\parallel - T_\perp\right) \qquad \frac{\mathrm{d}}{\mathrm{d}t}T_\perp = -\frac{1}{2}\nu_p\left(T_\perp - T_\parallel\right)$$

C) High-frequency plasma conductivity and collisional absorption

Let us assume plasma with constant electron density n_e and with neutralizing background of nonmoving ions. Electric field of electromagnetic wave is treated dipole approximation $\vec{E} = \vec{E}_0 \cos \omega t$. Action of magnetic force can be neglected for nonrelativistic intensities.

Kinetic equation for electrons then reads as follows

$$\frac{\partial f_e}{\partial t} - e\vec{E}\frac{\partial f_e}{\partial \vec{p}} = A\frac{\partial}{\partial p_i}\left(\frac{p^2\delta_{ij} - p_i p_j}{p^3}\frac{\partial f_e}{\partial p_j}\right) + C_{ee}\left(f_e\right) \quad ,$$

where $q_e = -e$, $q_i = Ze$, $A = Ze^4 m_e n_e \ln\Lambda_{ei} / (8\pi\varepsilon_0^2)$ and the term $C_{ee}(f_e)$ represents mutual electron collisions. Mutual electron collisions affect the symmetric part of the distribution function, but they do not directly influence the asymmetric part caused by the electric field, and thus, they can be omitted in calculation of conductivity. Angular eigenfunctions of the operator of electron-ion collisions are spherical harmonics, the distribution function can be expanded into series of spherical harmonics. Let us assume electric field in z direction. For weak field, when oscillatory velocity is << thermal one, it is enough to assume

$$f_e\left(\vec{p}\right)\cong f_0\left(p\right)+f_1\left(p\right)\cos\theta = f_0\left(p\right)+\frac{p_z}{p}f_1\left(p\right)$$

Then linearized kinetic equation reads as follows

$$\frac{\partial f_1}{\partial t}-eE\frac{\partial f_0}{\partial p}=-\frac{2A}{p^3}f_1 .$$

While f_0 is slowly varying, E a f_1 oscillate with frequency ω.

We shall use complex notation $E=E_0\exp\left(-i\omega t\right)$ and then

$$f_1\left(p\right)=ieE_0\frac{\partial f_0}{\partial p}\left(\omega+\frac{i2A}{p^3}\right)^{-1}=\frac{eE_0}{\omega}\left(i+\frac{2A}{\omega p^3}\right)\frac{1}{1+\left(\frac{2A}{\omega p^3}\right)^2}\frac{\partial f_0}{\partial p}$$

We denote $g\left(p\right)=1/\left[1+\left(2A/\omega p^3\right)^2\right]$; function $g(p)\cong 1$ with exception of small neighborhood of point $p=0$. The imaginary part of the current leads to the contribution $-\omega_p^2/\left(\omega^2+\nu_{ei}^2\right)$ to the real part of permittivity. Real part of f_1 leads to absorption of electromagnetic wave and Joule heating. Absorbed power is then

$$P_{abs} = \left\langle \vec{j}\vec{E} \right\rangle = \frac{1}{2}\mathrm{Re}\left[E_0 \left(-e \right) \int \mathrm{v}_z \frac{p_z}{p} f_1 \left(p \right) \mathrm{d}^3 p \right] = -\frac{eE_0}{2m_e} \int \frac{p_z^2}{p} \frac{eE_0}{\omega} g\left(p \right) \frac{2A}{\omega p^3} \times$$

$$\times \frac{\partial f_0}{\partial p} \mathrm{d}^3 p = -\frac{e_2 E_0^2 A}{m_e \omega^2} \int_0^\infty \mathrm{d}p \int_0^\pi d\theta \cos^2\theta \frac{g\left(p \right)}{p^2} \frac{\partial f_0}{\partial p} 2\pi p^2 \sin\theta =$$

$$= -\frac{e_2 E_0^2}{m_e \omega^2} \frac{4\pi A}{3} \int_0^\infty \mathrm{d}p \, g\left(p \right) \frac{\partial f_0}{\partial p} \cong \frac{4\pi A}{3} \frac{e_2 E_0^2}{m_e \omega^2} f_0 \left(p = 0 \right) \quad .$$

Collisional (inverse bremsstrahlung) absorption has only negligible influence on the shape of the distribution function f_0, if electron-electron collisions are frequent enough $\nu_{ee} \mathrm{v}_{Te}^2 \gg \nu_{ei} \mathrm{v}_{osc}^2$. This condition is equivalent to the condition $Z\, \mathrm{v}_{osc}^2 \ll \mathrm{v}_{Te}^2$. This condition need not be met for many times ionized plasma even if our original assumption $\mathrm{v}_{Te}^2 \gg \mathrm{v}_{osc}^2$ holds. In such case, collisional absorption leads to non-Maxwellian electron distribution. If there holds $Z\, \mathrm{v}_{osc}^2 \gg \mathrm{v}_{Te}^2$, electron-electron collision may be entirely omitted. If f_{1R} denotes real part of f_1 and $\langle\rangle$ averaging over angles, then

$$\frac{\partial f_0}{\partial t} = \frac{1}{2}\left\langle eE_0 \frac{\partial}{\partial p_z}\left[\frac{p_z}{p} f_{1R}(p)\right]\right\rangle_\theta = \frac{e E_0}{6p^2}\frac{\partial}{\partial p}\left(p^2 f_{1R}\right)$$

When using the approximation $g(p) \cong 1$ and substituting for $f_{1\mathrm{R}}$, the equation has the following form

$$\frac{\partial f_0}{\partial t} = \frac{e^2 E_0^2 A}{\omega^2}\frac{1}{3p^2}\frac{\partial}{\partial p}\left(\frac{1}{p}\frac{\partial f_0}{\partial p}\right).$$

We search for solution that does not change its form in time (self-similar solution). The solution is assumed in the following form

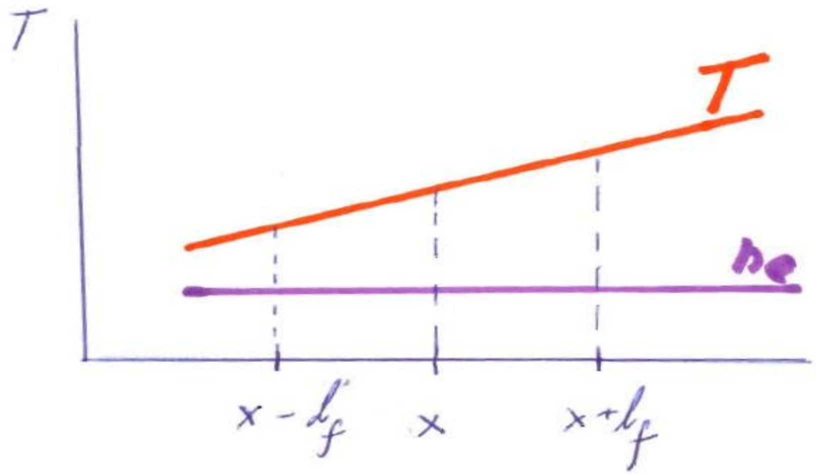
$f_0(p) = \frac{C_\alpha}{q^3(t)}\exp\left(-\frac{p^\alpha}{\alpha q^\alpha(t)}\right)$. One finds that $\alpha = 5$ and $C_5 = 5^{2/5} n_e / \Gamma(7/5)$.

Thermal momentum $q(t)$ is thus described by the following equation

$\frac{\mathrm{d}q}{\mathrm{d}t} = \frac{e^2 E_0^2}{\omega^2}\frac{A}{3q^4}$ with solution $q = \left(\frac{5e^2 E_0^2 A}{3\omega^2} t\right)^{1/5}$ and $k_B T_e = \frac{4\pi\, 5^{4/5}}{3^{7/5}\Gamma(7/5) m_e}\left(\frac{e^2 E_0^2 A}{\omega^2} t\right)^{2/5}$

Solution of equation for f_0 converges asymptotically to this shape from any initial distribution. This distribution function modification is called the Langdon effect - A.B. Langdon, Phys. Rev. Lett. **44**, 575 (1980).

D) Electron thermal conductivity of plasmas



Ion heat flux can be omitted due to small ion thermal velocities. Simple illustrative procedure will enable us to determine the electron thermal conductivity with accuracy up to a constant of order 1. Plasma with constant density and with temperature gradient is assumed. According to a crude picture, positive flux is caused by electrons moving from place in distance of the mean free path l_f back from the point x and negative flux is carried by electrons from $x+l_f$. Then

$$Q \cong Q_+\left(x-l_f\right)-Q_-\left(x+l_f\right) \cong {}^1\!/_6\, n \mathrm{v}_T\left(x-l_f\right) {}^3\!/_2\, k_B T_e\left(x-l_f\right)-{}^1\!/_6\, n \mathrm{v}_T\left(x+l_f\right)\times$$

$$\times\, {}^3\!/_2\, k_B T_e\left(x+l_f\right) \cong -\frac{1}{4} n k_B 2 l_f \frac{\partial}{\partial x}\left(\mathrm{v}_T T_e\right)=-\frac{3}{4} n \frac{k_B^{3/2} l_f T_e^{1/2}}{m_e^{1/2}} \frac{\partial}{\partial x} T_e \sim -T_e^{5/2} \frac{\partial}{\partial x} T_e$$

and flux Q does not depend on density, as mean free path $l_f = \mathrm{v}_T / \nu \sim T_e^2 / n$.
This derivation also demonstrates that the usual expression for thermal flux via temperature gradient is only the first term of Taylor expansion, and thus it holds only, if the temperature scale length L_T is >> mean free path l_f.
Now we derive thermal flux from Fokker-Planck equation. Ions are assumed nonmoving and we assume that current has to be zero for quasineutrality conservation. Thus, heat flux will be accompanied by the formation of an electric field. Let us assume temperature gradient in x direction. Then

$$f \cong f_M(x,p) + \frac{p_x}{p} f_1(x,p) = \frac{n_e}{\left(2\pi m_e k_B T_e(x)\right)^{3/2}} \exp\left(-\frac{p^2}{2m_e k_B T_e(x)}\right) + \frac{p_x}{p} f_1(x,p)$$

Then following equation holds for f_1

$$\mathrm{v}_x \frac{\partial f_M}{\partial x} - eE_x \frac{p_x}{p}\frac{\partial f_M}{\partial p} = -\frac{2A}{p^3}\frac{p_x}{p} f_1 + 2C^{ee}\left(f_M, f_1\right)$$

Again, we shall assume Z>>1 and we omit the impact of electron-electron collisions. Then f_1 is expressed as follows

$$f_1 = -\frac{p^4}{2Am_e} f_M \left[\frac{p^2}{2m_e k_B T_e} - \frac{3}{2}\right]\frac{1}{T_e}\frac{\mathrm{d}T_e}{\mathrm{d}x} - \frac{eE}{2A}\frac{p^4}{m_e k_B T_e} f_M$$

We obtain electric field E from requirement of zero electron flux

$$0=\int \mathrm{v}_x \frac{p_x}{p} f_1 \,\mathrm{d}^3 p=\frac{4\pi}{3m_e}\int_0^\infty p^3 f_1 \,\mathrm{d}p \qquad \Rightarrow \qquad \frac{eE}{m_e k_B T_e}=-\frac{5}{2m_e}\frac{1}{T_e}\frac{\mathrm{d}T_e}{\mathrm{d}x}$$

a function f_1 is thus

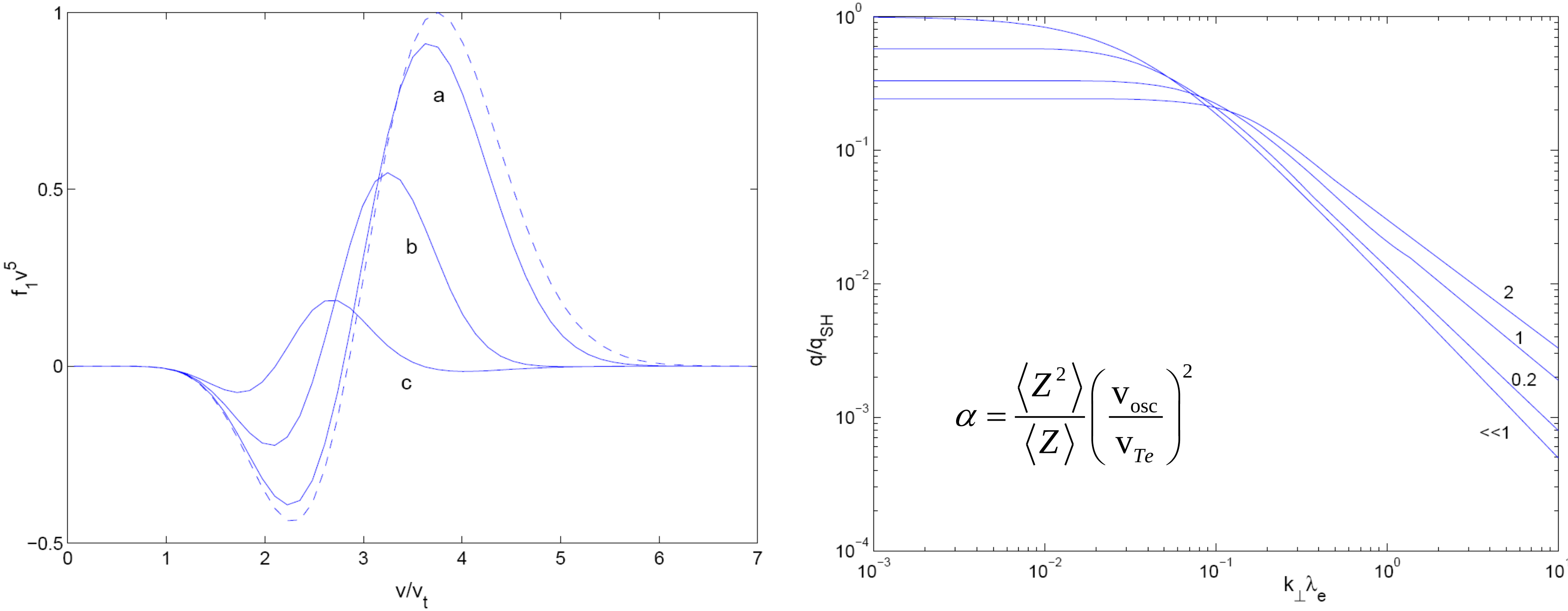
$$f_1=-\frac{p^4}{2Am_e} f_M\left[\frac{p^2}{2m_e k_B T_e}-4\right]\frac{1}{T_e}\frac{\mathrm{d}T_e}{\mathrm{d}x} \quad .$$

The heat flux Q is obtained after integration

$$Q=\int \frac{p^2}{2m_e}\mathrm{v}_x \frac{p_x}{p} f_1 \,\mathrm{d}^3 p=-\frac{4\pi\left(2\pi m_e k_B\right)^{7/2} n_e T_e^{5/2}}{Am_e}\frac{\mathrm{d}T_e}{\mathrm{d}x}=$$

$$=-\frac{\left(8\pi\varepsilon_0\right)^2\left(2\pi k_B\right)^{7/2} m_e^{3/2}}{Z\, e^4 \ln\Lambda_{ei}} T_e^{5/2}\frac{\mathrm{d}T_e}{\mathrm{d}x}=-\kappa_0\frac{\mathrm{d}T_e}{\mathrm{d}x}$$

The largest part of heat flux is carried by electrons with $\mathrm{v}\simeq 3\mathrm{v}_{Te}=3\sqrt{k_B T_e / m_e}$. Spitzer and Härm (Phys. Rev. **89** (1953), 977) calculated heat conductivity numerically including also electron-electron collision. Numerical result is approximated with a good accuracy by the formula $\kappa\cong\kappa_0\left(1+3.3/Z\right)^{-1}$.



Left panel - Distribution function $f_1 v^5$ relevant for electron heat flux for various $k_\perp \lambda_e$ ($k_\perp = 2\pi/L_T$, where L_T is temperature scale length; λ_e is the mean free path of electron with thermal velocity v_{Te}): $k_\perp \lambda_e = 0$ (dashed line – Spitzer-Harm thermal flux); $k_\perp \lambda_e = 0.01$ (a); $k_\perp \lambda_e = 0.05$ (b); $k_\perp \lambda_e = 0.2$ (c)

Right panel – The ratio of heat flux q to classic Spitzer-Harm heat flux q_{SH} in dependence on $k_\perp \lambda_e$ and on parameter α for collisional absorption of laser radiation. [power $m = 2 + 3/(1 + 1.67/\ \alpha^{0.724})$ in the distribution exponent]