Diffusion

Fick's law of diffusion in gas

Let's assume motionless ideal gas of constant temperature T and constant pressure p, in which there is a low inhomogeneous mass concentration of admixture α.
Then the total force on admixture α in stationary state must be zero

$$0 = -m_\alpha n_\alpha \nu_{\alpha p} \vec{u}_\alpha - \nabla p_\alpha = -m_\alpha n_\alpha \nu_{\alpha p} \vec{u}_\alpha - k_B T \nabla n_\alpha \ ,$$

where $\nu_{\alpha p}$ is collision frequency of admixture α with gas molecules.

The flux of particles α is $\vec{\Gamma}_\alpha = n_\alpha \vec{u}_\alpha = -D_\alpha \nabla n_\alpha = -\frac{k_B T}{m_\alpha \nu_{\alpha p}} \nabla n_\alpha$ ***Fick's law***

After substitution into continuity equation

$$\frac{\partial n_\alpha}{\partial t} = \mathrm{div}\left(D_\alpha \nabla n_\alpha\right) = D_\alpha \Delta n_\alpha$$

as the diffusion coefficient is constant here

Parabolic partial differential equation $L^2 = D\tau$

Suggested reading: Chen chap. 5, 8.2

Diffusion in weakly ionized gas

$n_n =>$ $\lambda_f = \dfrac{1}{n_n \sigma}$ $\nu = n_n \langle \sigma \mathrm{v} \rangle = \left\langle \dfrac{\mathrm{v}}{\lambda_f} \right\rangle$

Equations of motion (without or along magnetic field)

$$m_\alpha n_\alpha \left[\frac{\partial \vec{u}_\alpha}{\partial t} + \left(\vec{u}_\alpha \nabla \right) \vec{u}_\alpha \right] = q_\alpha n_\alpha \vec{E} - m_\alpha n_\alpha \nu_{\alpha n} \vec{u}_\alpha - \nabla p_\alpha$$

α is the particle type, stationary state. $\dfrac{\partial \vec{u}_\alpha}{\partial t} = 0$, term $\left(\vec{u}_\alpha \nabla \right) \vec{u}_\alpha$ omitted (quadratic)

$$\vec{u}_\alpha = \frac{q_\alpha}{m_\alpha \nu_{\alpha n}} \vec{E} - \frac{k_B T_\alpha}{m_\alpha \nu_{\alpha n}} \cdot \frac{\nabla n_\alpha}{n_\alpha} \qquad \vec{\Gamma}_\alpha = n_\alpha \vec{u}_\alpha = \pm \mu_\alpha n_\alpha \vec{E} - D_\alpha \nabla n_\alpha$$

μ_e, μ_i are mobilities $\mu_\alpha = \left| \dfrac{q_\alpha}{m_\alpha \nu_{\alpha n}} \right|$ D_e, D_i diffusion coefficients $D_\alpha = \dfrac{k_B T_\alpha}{m_\alpha \nu_{\alpha n}}$

Ambipolar diffusion – during diffusion electric field arises to secure quasineutrality

charge must stay ≈ 0 => $q_e\vec{\Gamma}_e + q_i\vec{\Gamma}_i = 0$ $\quad q_e n_e + q_i n_i \simeq 0$

Weakly ionized gas $Z = 1$ $\quad q_i = e \quad q_e = -e$ $\quad \Gamma_e = \Gamma_i = \Gamma$ $\quad n_e = n_i = n$

$$\vec{\Gamma} = \mu_i n\vec{E} - D_i \nabla n = -\mu_e n\vec{E} - D_e \nabla n \quad \Rightarrow \quad \vec{E} = \frac{D_i - D_e}{\mu_i + \mu_e}\frac{\nabla n}{n}$$

Electric field is proportional to the density gradient and ambipolar diffusion coefficient D_{a}

$$\vec{\Gamma} = -\frac{\mu_i D_e + \mu_e D_i}{\mu_i + \mu_e}\nabla n = -D_a \nabla n$$

In plasma without B is $\mu_e >> \mu_i$ and

$$D_a \approx D_i + \frac{T_e}{T_i}D_i \qquad (\approx 2D_i \quad \text{at equal temperatures of } e \text{ and } i)$$

Diffusion in direction perpendicular to B

In weakly ionized gas ν is collision frequency of studied particles with neutrals, let B is in direction of axis z and density gradient is in direction of x axis, and let $\tau = \nu^{-1}$

$$0 = -k_B T \frac{\partial n}{\partial x} + nqu_y B - \nu nmu_x \qquad 0 = -nqu_x B - \nu nmu_y$$

Then $$u_x = -\frac{k_B T}{nm\nu\left(1+\frac{\omega_c^2}{\nu^2}\right)}\frac{\partial n}{\partial x} \Rightarrow \Gamma_\perp = -\frac{k_B T}{m\nu\left(1+\omega_c^2\tau^2\right)}\frac{\partial n}{\partial x} = -\frac{D_\parallel}{\left(1+\omega_c^2\tau^2\right)}\frac{\partial n}{\partial x}$$

$$D_\perp = \frac{D_\parallel}{1+\omega_c^2\tau^2}$$

If $\omega_c\tau = \frac{\omega_c}{\nu} = \frac{\lambda_f}{r_L} \gg 1$ then $D_\perp = \frac{D}{\omega_c^2\tau^2} = \frac{k_B T\nu}{m\omega_c^2} = \frac{mk_B T\nu}{q^2 B^2}$

Diffusion coefficient across B is directly proportional to collision frequency, without collisions there would be no diffusion, after collision particle moves by 2 Larmor radiuses at maximum, thus the Larmor radius (gyroradius) substitutes the mean free path.

Across B ions are more mobile than electrons – electric field generated during ambipolar diffusion accelerates electrons and slows ions

Additionally, particle flux arises in the direction normal to B and ∇n, it is diamagnetic drift that we derived earlier (chapter 4) disregarding collisions ($\omega_c \tau \gg 1$).

In **fully ionized plasma** collision term for electrons will be $\nu_{ei} n_e m_e \left(\vec{u}_i - \vec{u}_e \right)$, diffusion velocities of electrons and ions will be equal, ambipolar field does not occur (and morever field in the direction of density gradient does not influence diffusion velocity in this direction)

$$D_\perp = \frac{k_B \left(T_e + T_i / Z \right) \nu_{ei}}{m_e \omega_c^{\ 2}} = \frac{m_e k_B \left(T_e + T_i / Z \right) \nu_{ei}}{e^2 B^2} \sim n T^{-1/2} B^{-2}$$

Classic collisional diffusion is proportional to B^{-2}, however, diffusion during magnetic confinement (in tokamaks etc.) is often higher, proportional to B^{-1}

Coefficient of **Bohm diffusion** was deduced from experimental results $D_\perp \simeq \frac{1}{16}\frac{kT_e}{eB}$

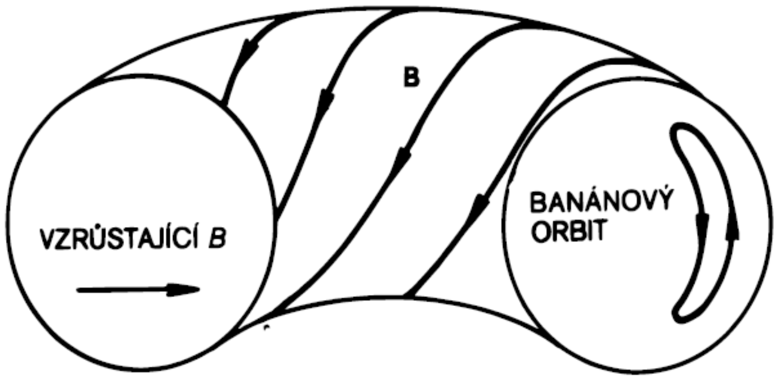
Various explanations- 1) Defects of magnetic field– possibility of field lines going to wall

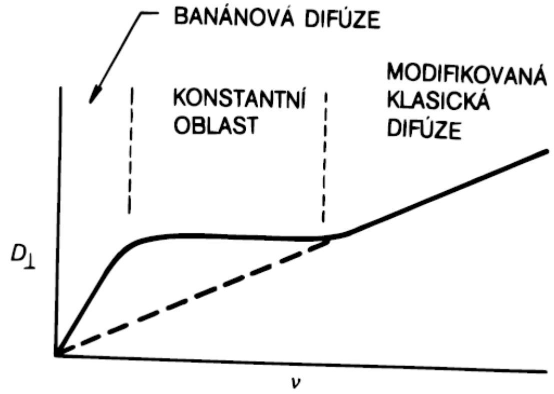
2) Asymmetric electric field – asymmetry of vacuum chamber or asymmetry of plasma generation or heating ⇒ *E*x*B* drift - convective cells

3) Instabilities leading to generation of plasma waves that produce oscillating electric field and *E*x*B* drifts

Accurate configuration of fields may bring the diffusion closer to the classic collisional limit.

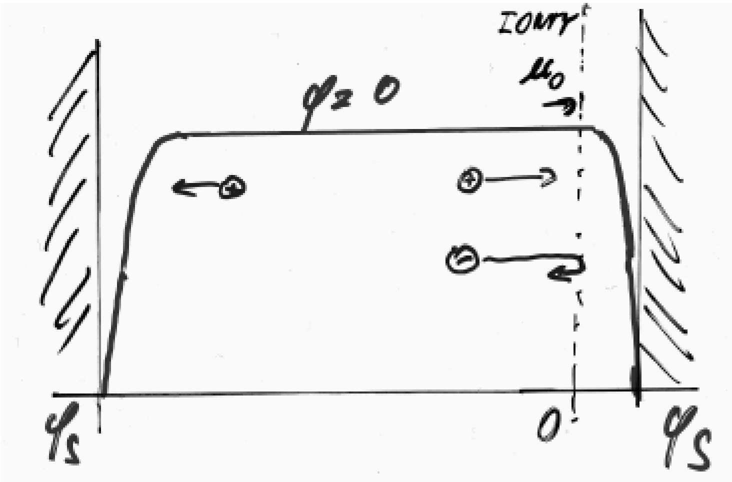
Diffusion can be increased in toroidal magnetic vessels due to existence of elongated closed orbits („banana orbit“) and this is called **neoclassical diffusion**.





Areas without quasineutrality

Sheath (near-wall layer)



Recombination on walls

Σ charge flux on walls = 0

Ions – assumed cold with velocity u_0 for $\varphi = 0$

$$u = \sqrt{{u_0}^2 - \frac{2e\varphi}{m_i}} \qquad \Big| \; \varphi < 0$$

$$n_i\left(x\right) = \frac{n_0 u_0}{u\left(x\right)} = \frac{n_0}{\sqrt{1 - \frac{2e\varphi}{m_i {u_0}^2}}}$$

Electrons – in equilibrium with field

$$n_e\left(x\right) = n_0 \exp\left(\frac{e\varphi}{k_B T_e}\right)$$

Poisson equation

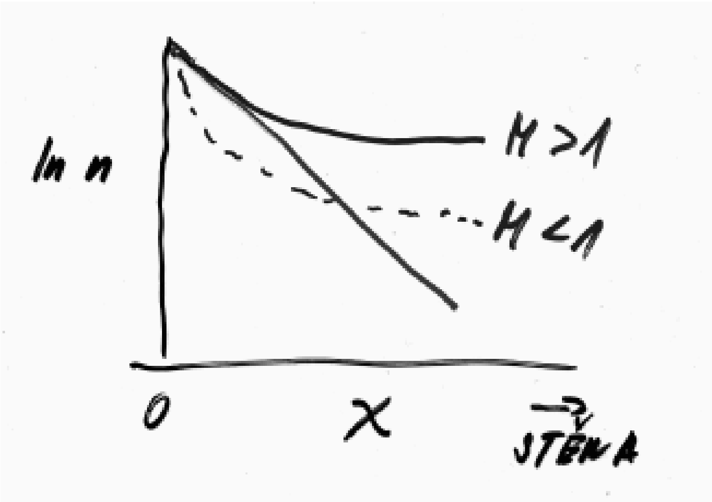
$$\frac{\mathrm{d}^2\varphi}{\mathrm{d}x^2} = \frac{e}{\varepsilon_0}\left(n_e - n_i\right) = \frac{en_0}{\varepsilon_0}\left[\exp\left(\frac{e\varphi}{k_B T_e}\right) - \frac{1}{\sqrt{1-\frac{2e\varphi}{m_i u_0}}}\right]$$

is transformed to dimensionless coordinates

$$\chi = -\frac{e\varphi}{k_B T_e} \qquad \xi = \frac{x}{\lambda_D} \qquad M = \frac{u_0}{\sqrt{k_B T_e / m_i}} = \frac{u_0}{c_s}$$

$$\chi'' = \left(1 + \frac{2\chi}{M^2}\right)^{-1/2} - e^{-\chi}$$ we multiply equation $\times\chi'$ and integrate $\int_0^\xi$

$$\frac{1}{2}\left(\chi'^2 - \chi_0'^2\right) = M^2\left[\left(1+\frac{2\chi}{M^2}\right)^{1/2} - 1\right] + e^{-\chi} - 1$$ in point 0 is $E \approx 0$ and thus $\chi_0' \approx 0$



Sheath keeps away electrons and attracts ions

$\Rightarrow \quad n_i > n_e$

Plane neighborhood, where plasma enters sheath, is derived from Taylor expansion for $\chi \approx 0$ terms proportional to 1 and χ cancel, the first ~ χ^2

$$\frac{1}{2}\chi'^2 \approx \frac{1}{2}\chi^2\left(-\frac{1}{M^2}+1\right) > 0$$

possible only for M > 1

Bohm criterion – stationary solution exists only for $u_0 > c_s$

How to find the potential φ_s of the wall?

$$\frac{1}{2}\mathrm{v}_{T_e} \cdot n_0 \exp\left(\frac{e\varphi_s}{k_B T_e}\right) \approx n_0 u_0 \qquad \varphi_s \approx \frac{k_B T_e}{e}\ln\frac{2u_0}{\mathrm{v}_{T_e}}$$

In the wall neighborhood $n_e << n_i,\ 2\chi >> M^2 \quad \Rightarrow \quad \chi'' = \left(1+\frac{2\chi}{M^2}\right)^{-1/2} \approx \frac{M}{\sqrt{2\chi}}$

after integration (z denotes place where one can assume $n_e \approx 0$)

$$\frac{1}{2}\left(\chi'^2 - \chi_z'^2\right) = \sqrt{2}M\left(\chi^{1/2} - \chi_z^{1/2}\right)$$

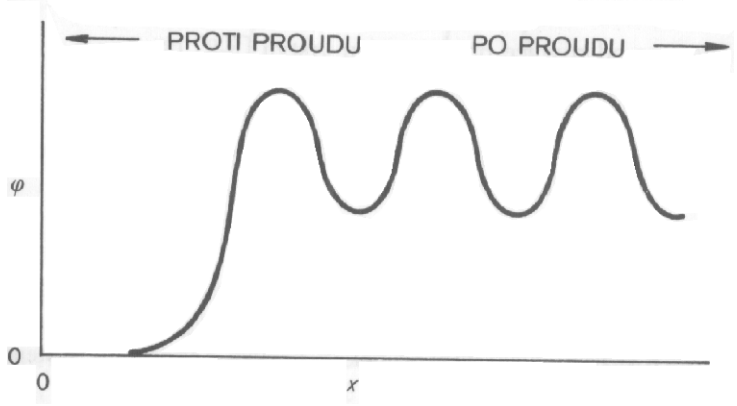
on the assumption $\chi_z \approx 0$, $\chi_z' \approx 0$ $\qquad \chi' = 2^{3/4} M^{1/2} \chi^{1/4}$

wall position is $\xi_s = \xi_z + \xi_d$ (ξ_d is the layer thickness $n_e \approx 0$)

then the relation holds $M = \frac{4\sqrt{2}}{9}\frac{\chi_s^{3/2}}{\xi_d^{2}} \quad \Rightarrow \quad J = en_0u_0 = \frac{4}{9}\left(\frac{2e}{m_i}\right)^{1/2}\frac{\varepsilon_0 \left|\varphi_s\right|^{3/2}}{d^2}$

Child-Langmuir law

Similar sheath is



collisionless ion-acoustic shock wave
(collisionless shock)

Sagdeev’s potential