Ion-sound waves (electrostatic low frequency waves)

are longitudinal waves similar classical sound

in gas $$c_s = \frac{\omega}{k} = \left(\frac{\gamma k_B T}{M}\right)^{1/2}$$

plasma – sound is slow for electrons, but fast for ions

Electron density is in each moment in equilibrium with instantaneous potential:

$$n_e = n_0 \exp\left(\frac{e\Phi_1}{k_B T_e}\right) = n_0\left(1 + \frac{e\Phi_1}{k_B T_e} + \dots\right)$$

$$n_1 = n_0 \frac{e\Phi_1}{k_B T_e}$$ electrons are isothermal ($\gamma = 1$)

Electric field may be also derived from quasi-neutrality condition–sum of forces acting on electrons = 0

$$eE_1 = -e\nabla\Phi_1 = -\frac{1}{n_0}\nabla p_{e1} = -\frac{k_B T_e}{n_0}\nabla n_1 = -ikk_B T_e \frac{n_1}{n_0} \qquad \Rightarrow E_1$$

hydrodynamic equation for ion fluid

$$i\omega n_{i1} = n_{i0} ik \mathrm{v}_{i1}$$

$$n_{i0} = \frac{n_0}{Z}$$

$$-i\omega M \mathrm{v}_{i1} = -\frac{\nabla p_{i1}}{n_{i0}} + ZeE_1$$

$$E_1 = -\nabla\Phi = -ik\Phi_1$$

Poisson equation is not needed

~~$-\varepsilon_0 \Delta\Phi_1 = Zen_{i0} - en_1 = e\left(Zn_{i1} - n_1\right)$~~

slow motion → quasineutrality $Zn_{i1} = n_1$

we express $\Phi_1 = \frac{k_B T_e}{en_0} n_1 = \frac{k_B T_e}{en_0} Zn_{i1}$ ⇒ equations of motion

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$$-i\omega M n_{i0} \mathrm{v}_{i1} = -ik\gamma_i k_B T_i n_{i1} - ikk_B T_e Zn_{i1}$$ - ion motion

$$i\omega n_{i1} = n_{i0} ik \mathrm{v}_{i1}$$ continuity equation

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$\Rightarrow$dispersion equation

$$\omega^2 = \underbrace{\left(\frac{\gamma_i k_B T_i + Z k_B T_e}{M} \right)}_{c_s{}^2} \cdot k^2$$

usually ions are adiabatic $\gamma_i = 5/3$

If $ZT_e \approx T_i$, then there is strong collisionless damping by ions, phase velocity $c_s \approx$ ion thermal velocity

Ion-sound waves for $ZT_e >> T_i$ are weakly damped

ion-sound velocity $$c_s \simeq \sqrt{\frac{Z k_B T_e}{M}}$$

The above applied plasmatic approximation does not hold for large k comparable with λ_{De}^{-1}. Consequently, dispersion relation will be derived without plasmatic approximation:

$$\nabla \vec{E}_1 = -\Delta \Phi_1 = k^2 \Phi_1 = e\left(Z n_{i1} - n_{e1}\right)/\varepsilon_0$$

$$n_{e1} = \frac{e\Phi_1}{k_B T_e} n_0$$

Density perturbation is included into Poisson equation

$$\Phi_1 \left(k^2 + \frac{n_0 e^2}{\varepsilon_0 k_B T_e} \right) = \frac{Zen_{i1}}{\varepsilon_0}$$

$$\Phi_1 = \frac{Zen_{i1}}{\varepsilon_0} \frac{\lambda_{De}{}^2}{1 + k^2 \lambda_{De}{}^2}$$

potential Φ_1 is included into ion equation of motion

$$\frac{\omega}{k} = \left(\frac{Zk_B T_e}{M} \frac{1}{1 + k^2 \lambda_{De}{}^2} + \frac{\gamma_i k_B T_i}{M} \right)^{1/2}$$

Dispersion relation of ion-sound waves differs due to inclusion of deviation from quasineutrality only by the term $k^2 \lambda_{De}^{\ \ 2}$,

The most simple relation $k >> \lambda_{De}^{\ \ -1}$ a $T_i = 0$

$$\omega^2 = \frac{n_0 Z e^2}{\varepsilon_0 M} = \frac{n_{i0} Z^2 e^2}{\varepsilon_0 M} = \omega_{pi}^{\ \ 2}$$

… ion plasma frequency

Elmg. waves in plasmas without external magnetic field B_0

Maxwell's equations — we transform to wave equation

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\frac{1}{\mu_0}\nabla \times \vec{B} = \varepsilon_0 \frac{\partial \vec{E}}{\partial t} + \vec{j}$$

$$\nabla \times \left(\nabla \times \vec{E}\right) + \varepsilon_0 \mu_0 \frac{\partial^2 \vec{E}}{\partial t^2} = \mu_0 \frac{\partial \vec{j}}{\partial t}$$

Plasma impact via high-frequency current density $\vec{j}$ (electron current dominant)
Fluid equations for electrons – equation for electron fluid velocity

$$\frac{\partial \vec{v}_e}{\partial t} + \left(\vec{v}_e \nabla\right) \vec{v}_e = -\frac{1}{n_e m_e} \nabla p_e - \frac{e}{m_e}\left(\vec{E} + \vec{v}_e \times \vec{B}\right)$$

A linear (i.e. weak) electromagnetic wave assumed, linearization

$$\vec{E} = \vec{E}_1 \quad \vec{B} = \vec{B}_1 \quad \vec{j} = \vec{j}_1 \quad \vec{v}_e = \vec{v}_1 \quad n_e = n_0 + n_1 \quad p_e = p_0 + p_1$$

Magnetic force $-e\left(\vec{v}_1 \times \vec{B}_1\right)$ omitted (small of the 2nd order)

$$\vec{v}_1 \| \vec{E}_1, \quad \mathrm{div}\vec{E}_1 = 0 \Rightarrow \mathrm{div}\vec{v}_1 = 0, \; n_1 = 0 \Rightarrow p_1 = 0$$

electron density does not change, equation for electron velocity simplified

$$\frac{\partial \vec{v}_{e1}}{\partial t} = -\frac{e}{m_e}\vec{E}_1$$

current density

$$\vec{j}_1 = -en_0\vec{v}_1 = -en_e\vec{v}_1 \quad \Rightarrow \quad \frac{\partial \vec{j}_1}{\partial t} = -en_e\frac{\partial \vec{v}_1}{\partial t} = \frac{e^2 n_e}{m_e}\vec{E}_1$$

For a monochromatic wave of frequency ω with time dependence exp(-iωt)

$$\vec{v}_1 = -\frac{ie\vec{E}_1}{m_e\omega} \qquad \vec{j}_1 = \sigma\vec{E}_1 = \frac{ie^2 n_e}{m_e\omega}\vec{E}_1$$

Relative permittivity for the electromagnetic (transverse) wave ε_r^{tr}

$$\varepsilon_r^{tr} = 1 + \frac{i\sigma}{\varepsilon_0\omega} = 1 - \frac{e^2 n_e}{\varepsilon_0 m_e \omega^2} = 1 - \frac{\omega_{pe}^2}{\omega^2}$$

(no spatial dispersion)

When expression for current is inserted into wave equation

$$-\Delta \vec{E}_1 + \frac{1}{c^2}\frac{\partial^2 \vec{E}_1}{\partial t^2} = -\mu_0 \frac{\partial \vec{j}_1}{\partial t} = -\frac{1}{c^2}\frac{e^2 n_e}{\varepsilon_0 m_e}\vec{E}_1$$

The dispersion equation is obtained for a wave of the wavevector $\vec{k}$

$$\omega^2 = \omega_{pe}^2 + c^2 k^2$$

When ion current is added, then $\omega_{pe}^2 \rightarrow \omega_p^2$

$$k^2 = \omega^2 \mu_0 \varepsilon_0 \varepsilon_r^{tr} = \omega^2 (1 - \omega_p^2 / \omega^2) / c^2$$

phase $v_\varphi = \dfrac{\omega}{k} = \sqrt{c^2 + \omega_p^2 / k^2}$ group $v_g = \dfrac{d\omega}{dk} = c^2 k / \omega = c^2 / v_\varphi$

for $\omega < \omega_p \rightarrow k^2 < 0$ wave does not propagate, it penetrates into plasma only via skin-effect

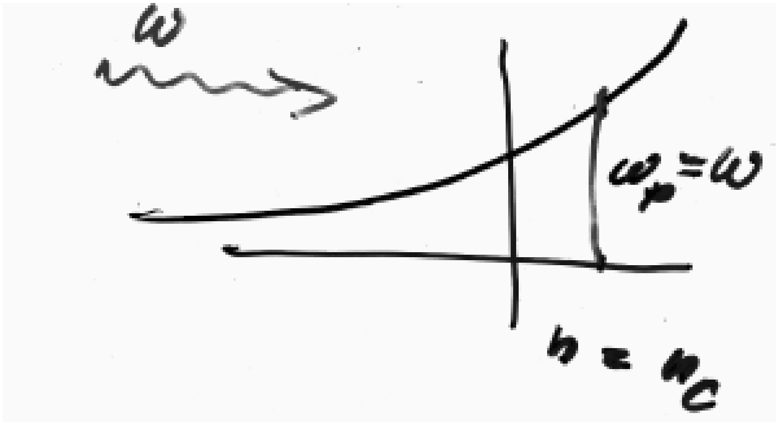
for $\omega \rightarrow \omega_{p+}$ $k \rightarrow 0$ and wave is totally reflected (cut off frequency)

!! no absorption in collisionless approximation

$$n_c = \frac{\varepsilon_0 m_e \omega^2}{e^2}$$

$$\mathrm{Re}(\varepsilon_r) = 1 - \frac{n_e}{n_c}$$

n_c – critical density



$n_c = 10^{21}\,\mathrm{cm}^{-3}$ → λ=1,06 μm (Nd-laser)

$n_c = 10^{19}\,\mathrm{cm}^{-3}$ → λ=10,6 μm (CO_2-laser)

$n_c = 10^{13}\,\mathrm{cm}^{-3}$ → λ=1,06 cm (cm waves)

Collisional absorption (also called **inverse bremsstrahlung**)

Electron collisions with other type of particles cause absorption (dominance of electron-ion collisions is assumed for simplicity)
Absorption of a photon during collision with ion is **time-reversed** process to the bremsstrahlung emission.

Derivation of absorption coefficient from microscopic description – tedious.
Macroscopically - collisions ⇒ friction slowing down electron oscillations

When electron collides with ion, the ordered oscillation velocity of electron transforms to stochastic thermal velocity and wave must missing oscillation energy to the electron – and energy of wave is thus absorbed by electrons

Equation for velocity of electron fluid with electron-ion collision frequency ν_{ei}

$$\frac{\partial \vec{v}_1}{\partial t} + \nu_{ei}\vec{v}_1 = -\frac{e}{m_e}\vec{E}_1$$

For a monochromatic wave of frequency ω with time dependence exp(-iωt)

$$\vec{v}_1 = -\frac{ie\vec{E}_1}{m_e\left(\omega + i\nu_{ei}\right)}$$

$$\vec{j}_1 = \sigma\vec{E}_1 = \frac{ie^2 n_e}{m_e\left(\omega + i\nu_{ei}\right)}\vec{E}_1 = i\frac{\omega e^2 n_e}{m_e\left(\omega^2 + \nu_{ei}^2\right)}\vec{E}_1 + \frac{\nu_{ei} e^2 n_e}{m_e\left(\omega^2 + \nu_{ei}^2\right)}\vec{E}_1$$

Part of current in phase with the electric field ⇒ Joule heating, absorption

Absorbed power per unit volume

$$W = \left\langle \vec{j}_1 \cdot \vec{E}_1 \right\rangle = \frac{1}{2} \Re \left\{ \vec{j}_1 \cdot \vec{E}_1^{\star} \right\} = \frac{1}{2} \frac{\nu_{ei} e^2 n_e}{m_e \left(\omega^2 + \nu_{ei}^2 \right)} |\vec{E}_1|^2$$

Relative permittivity - complex with positive imaginary part

$$\varepsilon_r^{tr} = 1 + \frac{i\sigma}{\varepsilon_0 \omega} = 1 - \frac{e^2 n_e}{\varepsilon_0 m_e \omega \left(\omega + i\nu_{ei} \right)} \simeq 1 - \frac{\omega_p^2}{\omega^2 + \nu_{ei}^2} + i \frac{\nu_{ei}}{\omega} \frac{\omega_p^2}{\omega^2 + \nu_{ei}^2}$$

Wavevector is then given by the expression

$$k^2 = \frac{\omega^2}{c^2} \varepsilon_r^{tr} = \frac{\omega^2}{c^2} \left(1 - \frac{\omega_p^2}{\omega^2 + \nu_{ei}^2} + i \frac{\nu_{ei}}{\omega} \frac{\omega_p^2}{\omega^2 + \nu_{ei}^2} \right)$$

Imaginary part-decrease of electric field amplitude in propagation direction

Absorbed energy is transferred to the **electron thermal energy**, ions are not directly heated. As energy exchange between electrons and ions is slow process, electron temperature in corona of laser-produced plasmas is higher than ion temperature.

Normal incidence of electromagnetic wave on planar plasma

Permittivity depends only on x ($\varepsilon_r = \varepsilon_r(x)$) and wavevector is in x direction

$$\mathrm{curl}\, E + \frac{\partial B}{\partial t} = 0 \qquad \mathrm{curl\,curl} = \mathrm{grad\,div} - \Delta$$

$$\mathrm{curl}\, B - \mu_0 \frac{\partial D}{\partial t} = 0 \qquad \mathrm{div}\, \vec{D} = 0 = \varepsilon\, \mathrm{div}\, \vec{E} + \underbrace{\vec{E}\nabla\varepsilon}_{0} \qquad \Rightarrow \mathrm{div}\, \vec{E} = 0$$

if time scale of density variations is $\tau >> \omega^{-1}$ $\quad \varepsilon_r(x,t) \to \varepsilon_r(x)$ and

$$\Delta E - \mu_0 \varepsilon_0 \varepsilon_r \frac{\partial^2 E}{\partial t^2} = 0 \qquad \frac{\partial}{\partial t} \to -i\omega \qquad E \sim e^{-i\omega t}$$

$$\frac{\partial^2 E}{\partial x^2} + \frac{\omega^2}{c^2}\varepsilon_r E = 0 \quad \Rightarrow \quad k^2 = \frac{\omega^2}{c^2}\varepsilon_r$$ stationary wave equation

if $\lambda \left|\frac{\nabla\varepsilon}{\varepsilon}\right| << 1 \ \Rightarrow$ ε is slowly varying in space

WKB approximation

$$E = E_+(x)e^{i\int k\,\mathrm{d}x} + E_-(x)e^{-i\int k\,\mathrm{d}x} \qquad E'' = \left[\underbrace{-k^2E_+}_{0.\,\text{řád}} + \underbrace{2ik\frac{\partial E_+}{\partial x} + i\frac{\partial k}{\partial x}E_+}_{1.\,\text{řád}} + \underbrace{\frac{\partial^2 E_+}{\partial x^2}}_{2.\,\text{řád}}\right]e^{i\int k\,\mathrm{d}x} + \ldots$$

0. order $\quad -k^2E_+ = \dfrac{\omega^2}{c^2}\varepsilon_r E_+ \quad$ fulfilled

1.order $\quad 2ik\dfrac{\partial E_+}{\partial x} + i\dfrac{\partial k}{\partial x}E_+ = 0 \qquad E \sim k^{-1/2} \sim \varepsilon_r^{-1/4}$

$$E = \frac{E_{0+}}{\sqrt[4]{\varepsilon_r}}e^{i\int k\,dx} + \frac{E_{0-}}{\sqrt[4]{\varepsilon_r}}e^{-i\int k\,dx}$$

WKB solution (no reflection !!)

Such density (permittivity) profiles do exist where WKB solves the task exactly

Critical point neighborhood – $\varepsilon \to 0$ - WKB approximation is not valid
Solution is found for linear profile of electron density n_e
relative permittivity $\varepsilon_r = -ax + iS$ where S = ν_{ei}/ω in the critical surface

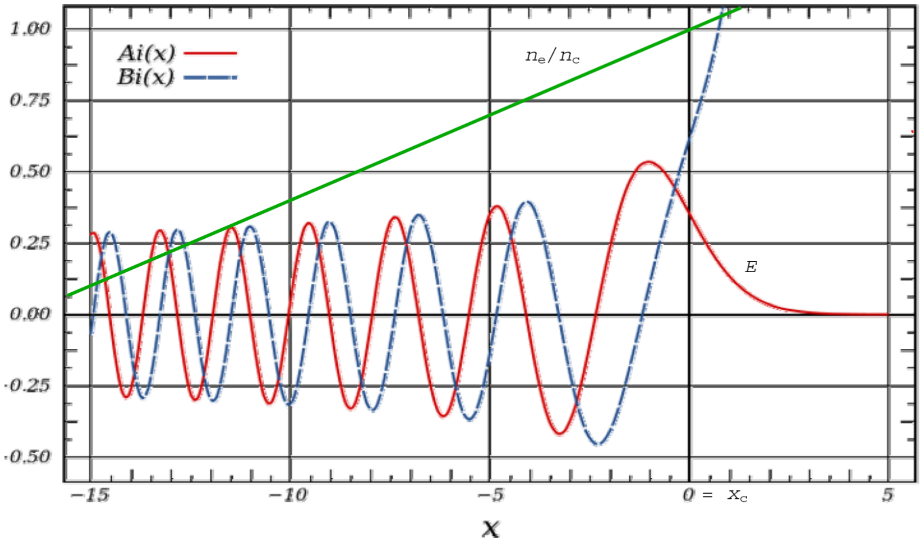
Plasma density is growing in direction of x axis – field must go to 0 for $x \to \infty$

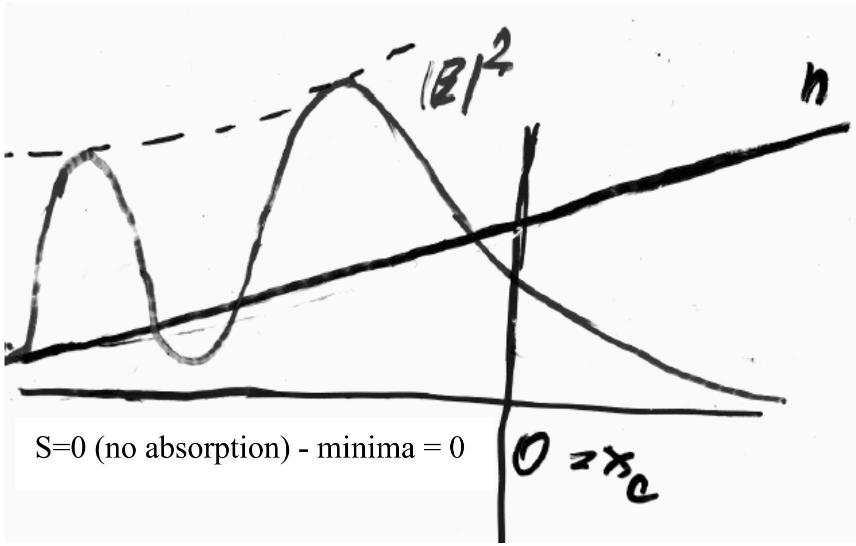
$$\frac{\partial^2 E}{\partial x^2} + \frac{\omega^2}{c^2}\left(-ax + iS\right)E = 0 \qquad \xi = \left(\frac{\omega}{ca}\right)^{2/3}\left(-ax + iS\right) \quad \Rightarrow \quad \frac{d^2 E}{d\xi^2} + \xi E = 0$$

There exists exact solution that fulfills the boundary condition

$E = 3C\,\mathrm{Ai}(-\xi)$ Ai = Airy function

$$= \begin{cases} C\xi^{1/2}\left[J_{1/3}\left(\frac{2}{3}\xi^{3/2}\right) + J_{-1/3}\left(\frac{2}{3}\xi^{3/2}\right)\right] & \mathrm{Re}(\xi) > 0 \\ C(-\xi)^{1/2}\left[I_{1/3}\left(\frac{2}{3}(-\xi)^{3/2}\right) + I_{-1/3}\left(\frac{2}{3}(-\xi)^{3/2}\right)\right] & \mathrm{Re}(\xi) < 0 \end{cases}$$





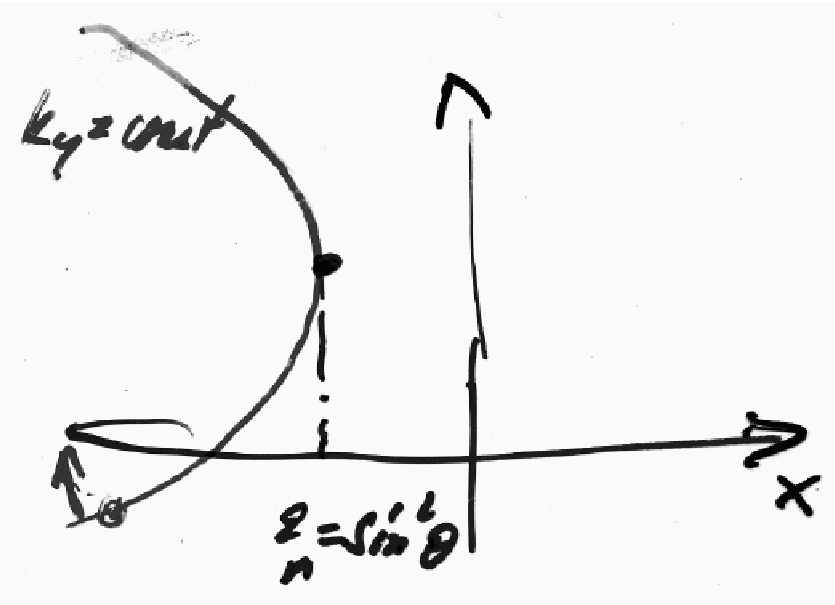
S=0 (no absorption) - minima = 0

Oblique incidence and resonance absorption

Permittivity depends only on x ($\varepsilon_r = \varepsilon_r(x)$) and wavevector $\vec{k} = (k_x, k_y, 0)$

$$k^2 = k_x^{\,2} + k_y^{\,2} = k_x^{\,2} + \frac{\omega^2}{c^2}\sin^2\theta_0$$

reflection point $\mathrm{Re}(\varepsilon_r) = \sin^2\theta_0$



$\odot E$, $\nwarrow B$ } TE wave = s-polarization

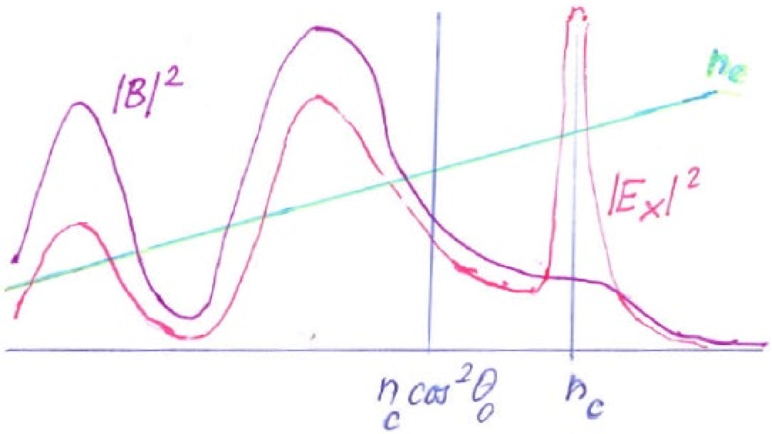
$\odot B$, $\nwarrow E$ } TM wave = p-polarization

p-polarization $\quad \vec{E}\cdot\nabla\varepsilon \neq 0 \quad \mathrm{div}\,\vec{E} \neq 0$

$$\frac{d^2B}{dx^2} - \frac{1}{\varepsilon}\frac{d\varepsilon}{dx}\frac{dB}{dx} + \frac{\omega^2}{c^2}\left(\varepsilon_r - \sin^2\theta_0\right)B = 0$$

$$E_x = -\frac{k_y B}{\omega \mu_0 \varepsilon} = -\frac{\sin\theta_0}{\sqrt{\mu_0 \varepsilon_0}} \cdot \frac{B}{\varepsilon_r}$$ singularity in the critical surface

<u>Resonance absorption</u> $t \rightarrow l$ (transverse elmg. wave transfers into longitudinal) - l – plasma wave cannot escape from plasma– collisional or collisionless absorption



in principle linear process – it exists even at small intensities I

when $\frac{\nu}{\omega} \rightarrow 0$ $A = f(q)$ $q = (k_0 L)^{2/3} \sin^2\theta_0$

$\theta_0 \rightarrow 0$ $E_x\big|_{x_c} \rightarrow 0$ normal incidence – no E_x

$L \rightarrow \infty$ $E_x\big|_{x_c} \rightarrow 0$ reflection point far from x_c

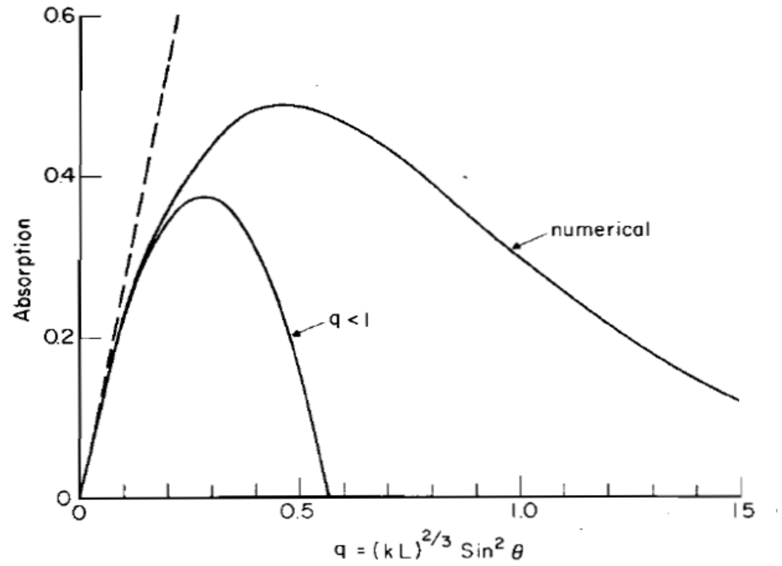
$$\frac{B(x_c)}{B_0} \cong \frac{2(\pi\cos\theta_0)^{1/2}}{3^{1/3}\,\Gamma(1/3)(k_0L)^{1/6}} \qquad \text{(for small } q\text{)}$$

$q \ll 1$

$$A = q\left(1-\frac{2}{3}q\right)$$

$q \gg 1$

$$A = 2\exp\left(-\frac{4}{3}q^{3/2}\right)-\exp\left(-\frac{8}{3}q^{3/2}\right)$$



maximum $A \approx 0.5$ when $q \approx 0.65$

collisions $\varepsilon(x_c) = i\dfrac{\nu_c}{\omega}$

$$E_z(x_c) = \frac{\sin\theta_0}{\frac{\nu_c}{\omega}} B(x_c)\cdot\frac{1}{\sqrt{\varepsilon_0\mu_0}}$$

The width of field maximum Δ

$$\underbrace{\frac{\left|n-n_c\right|}{n_c}}_{\frac{\Delta}{L}}=\frac{\nu_c}{\omega}\quad\Rightarrow\quad\Delta=\frac{\nu_c}{\omega}L$$

Absorbed energy

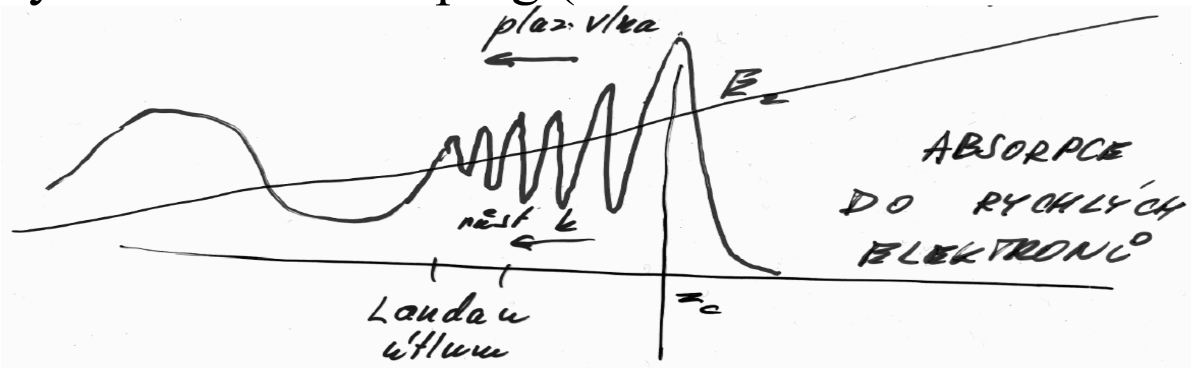
$$W=\frac{\omega}{c}\int_{-\Delta/2}^{\Delta/2}\frac{\nu_{ei}}{\omega}\left|E\right|^2dz\simeq\frac{\nu_c}{\omega}\frac{\omega}{c}\left|E_z\left(x_c\right)\right|^2\cdot\Delta=\frac{\omega}{c}\frac{\nu_c}{\omega}\frac{\sin^2\theta_0}{\left(\frac{\nu_c}{\omega}\right)^2}B^2\left(x_c\right)c^2\cdot\left(\frac{\nu_c}{\omega}\right)L$$

$W=\omega cB^2\left(x_c\right)\times L\sin^2\theta_0$ independent of ν_c

Warm plasma (spatial dispersion of longitudinal field)

$$\frac{1}{\varepsilon_0}\vec{D} = \varepsilon_r^{\;tr}\vec{E} + \frac{3\mathrm{v}_{Te}^{\;2}}{c^2}\left[\mathrm{grad}\ \mathrm{div}\vec{E} - \frac{1}{3}\frac{\nabla\varepsilon_r}{\varepsilon_r - 1}\mathrm{div}\vec{E}\right]$$

Plasma wave propagates from critical surface to rarified plasma, in decreasing density wavenumber k grows and thus v_φ decreases. When it becomes comparable to thermal velocity ⇒ Landau damping (it accelerates electrons out of plasma)



At higher intensities plasma wave damps via nonlinear mechanism of wavebreaking – energy is transferred to a small group of so called „hot (fast) electrons. Electrons are accelerated both into target, and to plasma-vacuum, where most of electrons are reflected in electrostatic field of sheath back into target.

Nonlinearities of electromagnetic wave propagation in plasmas

$$\varepsilon_r = 1 - \frac{e^2 n_e}{\varepsilon_0 m_e \omega^2} = 1 - \frac{\omega_p{}^2}{\omega^2}$$

A. $\rightarrow m_e$ ***– relativistic nonlinearity***

$$m_e = \frac{m_{e0}}{\sqrt{1 - \frac{\mathrm{v}^2}{\mathrm{c}^2}}} \approx \frac{m_{e0}}{\sqrt{1 - \frac{e^2 \left|E_L\right|^2}{2 m_e{}^2 c^2 \omega^2}}}$$

(for $\mathrm{v}_{osc} >> \mathrm{v}_{Te}$)

if $\mathrm{v}_{osc} << c \Rightarrow$

$$\varepsilon_r = 1 - \frac{e^2 n_e}{\varepsilon_0 m_{e0} \omega^2} \left(1 - \frac{1}{4} \frac{e^2 \left|E_L\right|^2}{m_{e0}^2 c^2 \omega^2} \right)$$

nonlinearity $\delta\varepsilon \sim \left|E_L\right|^2 / \omega^2 \sim I \lambda^2$ - quadratic nonlinearity – permittivity **grows** with the quadrat of electric field

B. $\rightarrow n_e$ ***- density modification caused by ponderomotive force or pressure gradient***

a) ponderomotive nonlinearity

$$F_p = -\frac{\rho}{2\rho_c}\varepsilon_0 \nabla \left\langle E^2 \right\rangle = -\frac{\rho}{4\rho_c}\varepsilon_0 \nabla \left|E_L\right|^2$$

$$F_p - \nabla p = 0$$

Ponderomotive force pushes plasma out of intense field $\Rightarrow$ it causes density gradient $\Rightarrow$ pressure gradient

In equilibrium pressure gradient compensates for ponderomotive force:

$$-\frac{n_e}{4n_c}\varepsilon_0 \nabla \left|E_L\right|^2 - k_B T_e \nabla n_e = 0 \quad \Rightarrow \quad n_e = n_0 \exp\left(-\frac{\varepsilon_0 \left|E_L\right|^2}{4 k_B T_e n_c}\right)$$

For small I – positive quadratic nonlinearity $\delta\varepsilon \sim \left|E_L\right|^2 / n_c \sim I\lambda^2$

b) thermal – plasma is most heated in the field maximum and the density decreases there to keep the pressure constant

Nonlinearity consequences

A. Formation of filaments (hot spots inside a laser beam)

Quadratic medium $\varepsilon_R = \varepsilon_L + \varepsilon_2 \left\langle E^2 \right\rangle$

Wave $E = E_0 \cos\left(k_0 x - \omega t\right)$ $\qquad E \parallel z \qquad \delta E \parallel z$

Perturbation (for simplicity periodic in the normal to *E* direction)

$$\delta E = \left[e_1\left(x\right)\cos\left(k_0 x - \omega t\right) + e_2\left(x\right)\sin\left(k_0 x - \omega t\right)\right]\cdot \cos(k_\perp y)$$

e_1 is the component in phase with the main planar wave

$$\varepsilon = \varepsilon_L + \varepsilon_2 \left\langle \left(E + \delta E\right)^2 \right\rangle = \underbrace{\varepsilon_L + \frac{1}{2}\varepsilon_2 E_0{}^2}_{\varepsilon_m} + \varepsilon_2 e_1\left(x\right) E_0 \cos(k_\perp y)$$

Field perturbation induces permittivity perturbation periodic in *y*

$$k_0{}^2 = \frac{\omega^2}{c^2}\varepsilon_m$$ wavevector in homogeneous medium

notation $$\Delta = \partial^2/\partial x^2 + (\partial^2/\partial y^2 + \partial^2/\partial z^2) = \nabla_\parallel^2 + \nabla_\perp^2$$

$$2ik_0\frac{\partial \delta E}{\partial x} + \nabla_\perp^2 \delta E + \frac{\omega^2}{c^2}\delta\varepsilon\, E_0 = 0$$ wave equation for perturbation

next rewritten for components

$$-2k_0\frac{de_2}{dx} - k_\perp{}^2 e_1 + \frac{\omega^2}{c^2}\varepsilon_2 E_0{}^2 e_1 = 0$$ ~cos (k_0x- ωt)

$$2k_0\frac{de_1}{dx} - k_\perp{}^2 e_2 = 0$$ ~sin (k_0x- ωt)

Assumption: $e_1(x) \sim e^{k_\parallel x}$ $$k_\parallel = \pm\frac{k_\perp}{2k_0}\sqrt{\frac{\omega^2}{c^2}\varepsilon_2 E_0{}^2 - k_\perp{}^2}$$

The higher is the radiation intensity, the narrower perturbations can grow.

Maximum growth $k_{\parallel} = \dfrac{\omega\,\varepsilon_2\,E_0^{\,2}}{4\,c\sqrt{\varepsilon_m}}$ for $k_{\perp}^{\,2} = \dfrac{\omega^2}{2c^2}\varepsilon_2 E_0^{\,2}$

Laser beam is finally split into high intensity zones (hot spots) – filaments are observed when looking from the side

B. Self-focusing

Laser beam focuses as a whole at lower intensity – filaments are formed earlier for high *I* and quadratic nonlinearity,

Wave equation (ε_L is linear and $\delta\varepsilon$ nonlinear part of permittivity)

$$\Delta E - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\left[\left(\varepsilon_L + \delta\varepsilon\right)E\right] = 0$$

$$\xi = t - \frac{x}{\mathrm{v}_g}$$

$$E = \Psi\left(x, y, z, \xi\right)\exp\left(ikx - i\omega\, t\right)$$

Ψ is slowly varying complex wave amplitude

$$\left(2ik\frac{\partial}{\partial x}+\nabla_\perp{}^2\right)\Psi=-k^2\frac{\delta\varepsilon}{\varepsilon_L}\Psi$$

parabolic equation

$k^2 = \omega^2 \varepsilon_L / c^2$

$$\Psi = A\exp(i\varphi)$$

Ψ is divided into real amplitude and phase

Then, one can write equations for the amplitude and the phase

$$k\frac{\partial}{\partial x}A^2=-\nabla_\perp\left(A^2\nabla_\perp\varphi\right)\ \ldots$$

describes the energy transport

$$\frac{\partial}{\partial x}\varphi+\frac{1}{2k}\left(\nabla_\perp\varphi\right)^2-\frac{k}{2}\left(\frac{\nabla_\perp{}^2 A}{k^2 A}+\frac{\delta\varepsilon}{\varepsilon}\right)=0$$

the shape of wavefront

diffraction (first term in parentheses), refraction (second term in parentheses)

For collimated beam $\nabla_\perp\varphi = 0$

Self-focusing of collimated beam occurs, if refraction is greater than diffraction

$$\frac{\delta\varepsilon}{\varepsilon} > -\frac{\nabla_\perp{}^2 A}{k^2 A}$$

For Gaussian beam

$$A = A_0 \exp\left(-\frac{r^2}{2a^2}\right)$$

In cylindric geometry

$$\nabla_\perp^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)$$

and self-focusing condition reads

$$\frac{\varepsilon_2 A_0{}^2}{\varepsilon_L} > \frac{2}{k^2 a^2}$$

As $A_0{}^2 a^2 \sim P$, self-focusing occurs, is beam power is $P > P_0$ – threshold power for self-focusing [for relativistic NL is $P_0 \simeq 17.5(n_c / n_e)$ GW].

Gaussian beam would focus according to the above equations ideally into a point focus, however the slowly varying amplitude approximation is not valid in its neighborhood.

Elmg. waves in plasma with external magnetic field B_0

Let's assume homogeneous stationary magnetic field B_0.
Plasma is **anisotropic**, we derive **conductivity tensor** σ_{ij} and from conductivity we shall calculate tensor of **high-frequency permittivity** ε_{ij}.
We shall assume high-frequency waves, and thus for simplicity we omit the ion current. Further, for simplicity we shall assume cold plasma and omit possible impact of gradient of high-frequency pressure.
Let axis z is in the direction of B_0. Let weak high frequency field E_1 (B_1 does not influence linear conductivity) is of frequency ω. Plasma is collisionless. Then

$$\vec{u}_1 = -i\frac{e}{m_e\omega}\vec{E}_1 - i\frac{\omega_c}{\omega}\vec{u}_1 \times \hat{z}$$

Magnetic field has no impact on conductivity in z direction and for directions x, y we obtain

$$u_{1x} = -i\frac{\omega e E_{1x}}{m_e\left(\omega^2 - \omega_c^2\right)} - \frac{e\,\omega_c E_{1y}}{m_e\left(\omega^2 - \omega_c^2\right)}$$

$$u_{1y} = \frac{e\,\omega_c E_{1x}}{m_e\omega^2\left(\omega^2 - \omega_c^2\right)} - i\frac{\omega e E_{1y}}{m_e\left(\omega^2 - \omega_c^2\right)}$$

The current is $\vec{j}_1 = -e\, n_{e0}\, \vec{u}_1 = \vec{\vec{\sigma}}\, \vec{E}_1$ and permittivity $\varepsilon_{ij} = \varepsilon_0\, \varepsilon_{r,ij} = \varepsilon_0 \left(\delta_{ij} + i \dfrac{\sigma_{ij}}{\omega\, \varepsilon_0} \right)$

$$\vec{\vec{\varepsilon}}_r = \begin{pmatrix} 1 - \dfrac{\omega_p^2}{\omega^2 - \omega_c^2} & i \dfrac{\omega_c}{\omega} \dfrac{\omega_p^2}{\omega^2 - \omega_c^2} & 0 \\ -i \dfrac{\omega_c}{\omega} \dfrac{\omega_p^2}{\omega^2 - \omega_c^2} & 1 - \dfrac{\omega_p^2}{\omega^2 - \omega_c^2} & 0 \\ 0 & 0 & 1 - \dfrac{\omega_p^2}{\omega^2} \end{pmatrix}$$

From Maxwell's equations one derives wave equation

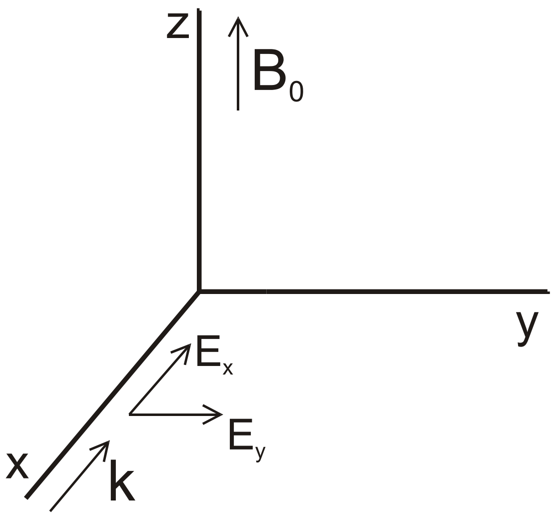
$$i\vec{k} \times \vec{E} - i\omega\vec{B} = 0 \qquad i\vec{k} \times \vec{B} + i\omega\varepsilon_0 \vec{\vec{\varepsilon}}_r \vec{E} = 0 \quad \Rightarrow \quad -\vec{k} \times \left(\vec{k} \times \vec{E} \right) - \frac{\omega^2}{c^2} \vec{\vec{\varepsilon}}_r \vec{E} = 0$$

Coordinate system is chosen so that $\vec{k} = \left(k_x, 0, k_z \right)$ a non-zero solution of homogeneous equation exists only if the determinant is equal to 0.

$$\det\begin{pmatrix} k_z^2-\frac{\omega^2}{c^2}\varepsilon_{r,xx} & -\frac{\omega^2}{c^2}\varepsilon_{r,xy} & -k_xk_z \\ -\frac{\omega^2}{c^2}\varepsilon_{r,yx} & k_x^2+k_z^2-\frac{\omega^2}{c^2}\varepsilon_{r,yy} & 0 \\ -k_xk_z & 0 & k_x^2-\frac{\omega^2}{c^2}\varepsilon_{r,zz} \end{pmatrix}=0$$

A. $k \perp B_0$ **(propagation normal to magnetic field)**

if $\vec{E} \parallel \vec{B}_0$ B_0 does not influence wave – **ordinary wave-O**



3rd line of determinant

$$k_x^2-\frac{\omega^2}{c^2}\varepsilon_{r,zz}=0 \quad \Rightarrow \quad \omega^2=\omega_p^2+c^2k^2$$

extraordinary wave - X

influenced by field B_0– 1st and 2nd row and column

=> $\mathrm{v}_{1y} \Rightarrow F_x \sim \mathrm{v}_{1y}\times B_0 \rightarrow \mathrm{v}_{1x} \Rightarrow j_x \Rightarrow E_x$

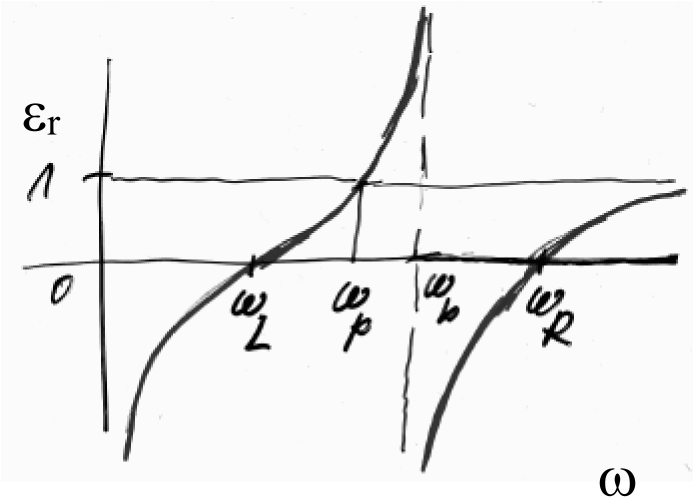
Electric field is not normal to the propagation direction

Dispersion relation for X wave is ($\omega_h^2 = \omega_p^2 + \omega_c^2$)

$$\frac{c^2k^2}{\omega^2} = \frac{c^2k_x^2}{\omega^2} = 1 - \frac{\omega_p^2}{\omega^2}\frac{\omega^2 - \omega_p^2}{\omega^2 - \omega_h^2}$$

Relative permittivity of plasma for X wave (indexu of refraction *n* squared)

$$\varepsilon_r = n^2 = 1 - \frac{\omega_p{}^2}{\omega^2}\frac{\omega^2 - \omega_p{}^2}{\omega^2 - \omega_h{}^2}$$



permittivity $\varepsilon_r = \pm\infty$ pro $\omega = \omega_h$

permittivity $\varepsilon_r = 0$ for $\omega_L = \frac{1}{2}\left[-\omega_c + \sqrt{\omega_c{}^2 + 4\omega_p{}^2}\right]$

and for $\omega_R = \frac{1}{2}\left[\omega_c + \sqrt{\omega_c{}^2 + 4\omega_p{}^2}\right]$

for $\omega \to \omega_{h-}$ there is resonance ($k \to \infty$) ⇒ total absorption of wave

points $\omega \to \omega_{L+}, \omega_{R-}$ are cutoffs ($k \to 0$)⇒ total reflection of the wave

A. $k \parallel B_0$ **(propagation along magnetic field)**

$$\det\begin{pmatrix} k_z^2 - \frac{\omega^2}{c^2}\varepsilon_{r,xx} & -\frac{\omega^2}{c^2}\varepsilon_{r,xy} \\ -\frac{\omega^2}{c^2}\varepsilon_{r,yx} & k_z^2 - \frac{\omega^2}{c^2}\varepsilon_{r,yy} \end{pmatrix} = 0$$

Left-handed circular wave (L wave)

$$\varepsilon_r = n^2 = 1 - \frac{{\omega_p}^2}{\omega^2}\frac{1}{1+\dfrac{\omega_c}{\omega}}$$

ω_L - cutoff, propagationfor $\omega > \omega_L$

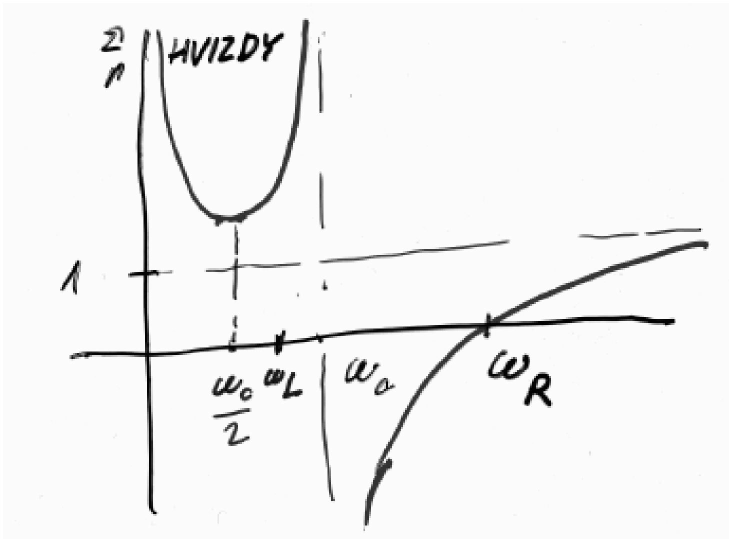
Right-handed circular wave (R wave) – cyclotron rotation of electrons is right-handed - resonance!!

$$\varepsilon_r = n^2 = 1 - \frac{{\omega_p}^2}{\omega^2}\frac{1}{1-\dfrac{\omega_c}{\omega}}$$

propagates for $\omega > \omega_R$

and $\omega < \omega_c$ (whistlers)– resonance for ω_c

permittivity for R wave



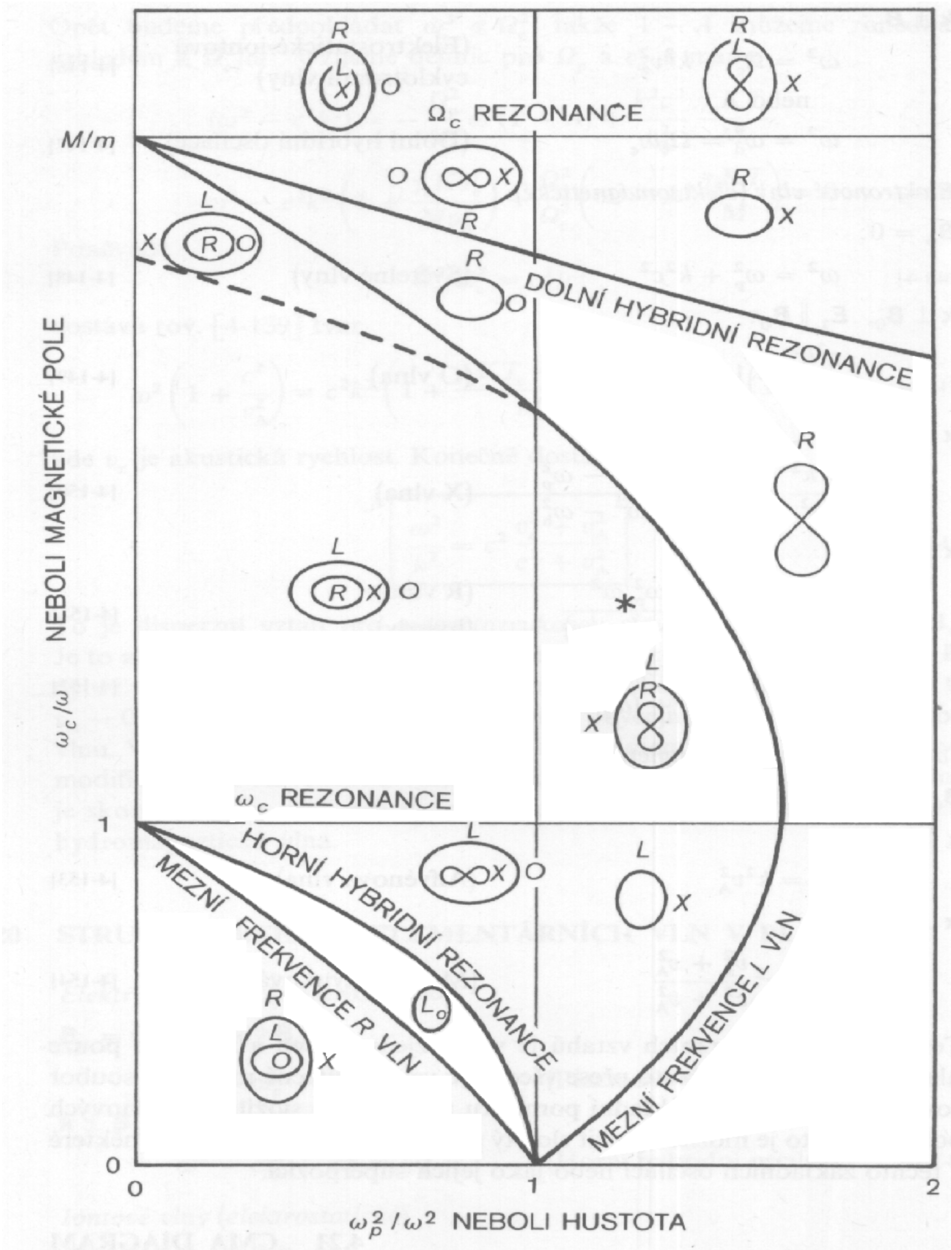
CMA diagram

(electromagnetic waves)

It contains

- boundaries between particular areas of propagation types
- vertically – propagation along B, horizontally – normally to B
- dependence of phase velocity on direction of propagation
- transition between types of waves if propagation direction changes

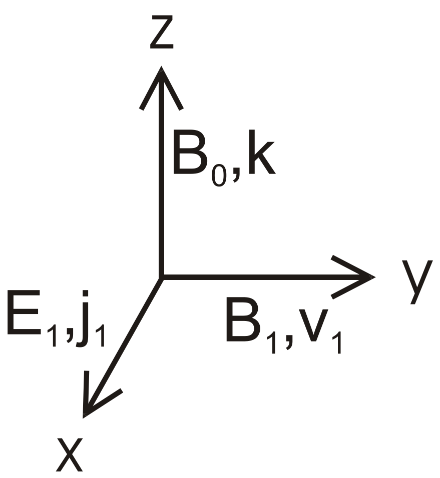
density (ω_p^2/ω^2) on horizontal axis magnetic field oriented along vertical axis – value (ω_c/ω) on vertical axis



Ion electromagnetic waves (hydromagnetic waves)

– exist only in presence of B_0

a) $\vec{k} \parallel \vec{B}_0$ – **Alfvén wave**



$$i\vec{k} \times \vec{E}_1 = i\omega \vec{B}_1$$

electron density

$$\frac{1}{\mu_0} i\vec{k} \times \vec{B}_1 + i\omega\varepsilon_0 \vec{E}_1 = n_0 e\left(\vec{v}_i - \vec{v}_e\right)$$

$$0 = -e\vec{E}_1 - e\vec{v}_e \times \vec{B}_0$$

$$-i\omega M_i \vec{v}_i = Ze\vec{E}_1 + Ze\vec{v}_i \times \vec{B}_0$$

equation of motion

$$v_{ex} = 0 \qquad v_{ey} = \frac{\vec{E}_1 \times \vec{B}_0}{B_0^{\,2}}$$

E×B drift

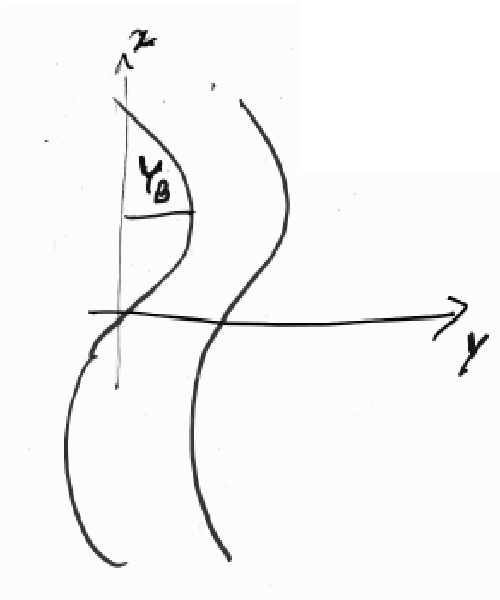
$$v_{iy} = -i\frac{ZeB_0}{\omega M_i} v_{ix} = -i\frac{\Omega_c}{\omega} v_{ix} \qquad \Omega_c = \frac{ZeB_0}{M_i}$$

ion cyclotron frequency

$$\mathrm{v}_{ix} = i\frac{ZeE_{1x}}{\omega M_i\left(1-\frac{\Omega_c^2}{\omega^2}\right)}$$

$$\omega_{pi}^2 = \frac{Ze^2 n_0}{\varepsilon_0 M_i}$$ ion plasma frequency

We substitute current into wave eq. ⇒dispersion rel. $1-\frac{k^2c^2}{\omega^2}+\frac{\omega_{pi}{}^2}{\Omega_c{}^2-\omega^2}=0$



for $\omega^2 \ll \Omega_c{}^2$ $\quad \omega^2 = \frac{k^2 \mathrm{v}_A{}^2}{1+\left(\frac{\mathrm{v}_A}{c}\right)^2}$ where $\mathrm{v}_A{}^2 = \frac{c^2\Omega_c{}^2}{\omega_{pi}{}^2} = \frac{c^2 B_0{}^2 \varepsilon_0}{\rho_M}$

Alfvén velocity

$$Y_B(z,t) = \int^z \frac{B_y(z',t)}{B_0}dz' \qquad \mathrm{v}_B = -i\omega Y_B = -\frac{\omega}{k}\frac{B_y}{B_0} \qquad B_y = \frac{k}{\omega}E_x$$

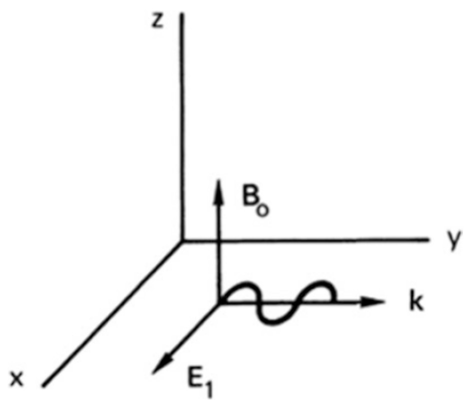
Y_B shift of field line position, v_B velocity of field line motion

$$\Rightarrow \quad \mathrm{v}_B = -\frac{E_x}{B_0} = \mathrm{v}_D$$

Plasma moves with the same velocity as the field line
Magnetic field line are „frozen” in plasma“

Alfvén wave – transverse waves „on string“ String ≅ magnetic field line
(Hannes Alfvén – Nobel prize – 1970)

b) $\vec{k} \perp \vec{B}_0$ – magnetosonic wave $\vec{E} \perp \vec{B}_0$



dispersion relation for cold plasma

$$\omega^2 = \frac{k^2 \mathrm{v_A}^2}{1+\frac{\mathrm{v_A}^2}{\mathrm{c}^2}} \approx k^2 \mathrm{v}_A^2$$

for $T_e \neq 0$ additional action of pressure

$$\omega^2 = k^2 \frac{\mathrm{v_A}^2 + c_s^2}{1+\frac{\mathrm{v_A}^2}{\mathrm{c}^2}} \approx k^2 \left(\mathrm{v}_A^2 + c_s^2\right)$$