**Foundations of kinetic theory – Klimontovich equation**
(suggested reading – D.R. Nicholson, *Introduction to plasma theory*, chap. 3)

phase space \((\tilde{x}, \tilde{v})\) - 6-dimensional space

density of 1 point particle \(\tilde{x}_1 (f)\), \(\tilde{V}_1 (t)\) - singular non-zero point

\[
N(\tilde{x}, \tilde{v}, t) = \delta[\tilde{x} - \tilde{X}_1 (t)] \delta[\tilde{v} - \tilde{V}_1 (t)]
\]

\(N_{0s}\) particles of sort „s“ (\(N_{0s}\) points)

\[
N_s (\tilde{x}, \tilde{v}, t) = \sum_{k=1}^{N_{0s}} \delta[\tilde{x} - \tilde{X}_k (t)] \delta[\tilde{v} - \tilde{V}_k (t)]
\]

Equations of motion – for particle number \(k\)

\[
\dot{\tilde{X}}_k (t) = \tilde{V}_k (t)
\]

\[
m_s \ddot{\tilde{V}}_k (t) = q_s E^m [X_k (t), t] + q_s \tilde{V}_k (t) \times \tilde{B}^m [X_k (t), t]
\]
Equations for fields – index $m = \text{microscopic field}$

\[
\begin{align*}
\text{div } \vec{E}^m &= \frac{\rho^m(x, t)}{\varepsilon_0} \\
\text{div } \vec{B}^m &= 0 \\
\text{rot } \vec{E}^m &= \frac{\partial \vec{B}^m}{\partial t} = 0 \\
\text{rot } \vec{B}^m &= \mu_0 \vec{J}^m + \varepsilon_0 \mu_0 \frac{\partial \vec{E}^m}{\partial t}
\end{align*}
\]

Microscopic charge and current densities

\[
\begin{align*}
\rho^m &= \sum_{e,i} q_s \int d\vec{v} \, N_s(\vec{x}, \vec{v}, t) \\
\vec{J}^m &= \sum_{e,i} q_s \int d\vec{v} \, \vec{v} \, N_s(\vec{x}, \vec{v}, t)
\end{align*}
\]

How evolves $N_s$ in time?

\[
N_s = \sum_{k=1}^{N_{0s}} \delta \left[ \vec{x} - \vec{X}_k(t) \right] \delta \left[ \vec{v} - \vec{V}_k(t) \right]
\]

\[
\frac{\partial N_s}{\partial t} = -\sum_{k=1}^{N_{0s}} \dot{X}_k \nabla_{\vec{x}} \delta \left[ \vec{x} - \vec{X}_k(t) \right] \delta \left[ \vec{v} - \vec{V}_k(t) \right] - \sum_{k=1}^{N_{0s}} \dot{V}_k \delta \left[ \vec{x} - \vec{X}_k(t) \right] \nabla_{\vec{v}} \delta \left[ \vec{v} - \vec{V}_k(t) \right]
\]
We shall substitute for temporal derivatives of particle position and velocity

\[
\frac{\partial N_s(\bar{x}, \bar{v}, t)}{\partial t} = -\sum_{k=1}^{N_0} \bar{V}_k \nabla_x \delta[\bar{x} - \bar{X}_k] \delta[\bar{v} - \bar{V}_k] - \\
- \sum_{k=1}^{N_0} \left\{ \frac{q_s}{m_s} E^m X_k(t), t] + \frac{q_s}{m_s} \bar{V}_k \times \bar{B}^m [X_k(t), t] \right\} \times \delta[\bar{x} - \bar{X}_k] \nabla_v \delta[\bar{v} - \bar{V}_k]
\]

Simple relation \( a\delta(a-b) = b\delta(a-b) \) is utilized

\[
\frac{\partial N_s}{\partial t} + \bar{v} \nabla_x N_s + \frac{q_s}{m_s} \left( \bar{E}^m + \bar{v} \times \bar{B}^m \right) \nabla_v N_s = 0
\]

**Klimontovich equation** *(around year 1960)*

- not usable for plasma description
- used for derivation of suitable equations
- it contains exact trajectories of all particles!
Total derivative along particle trajectory in the phase space – particle density along trajectory does not change
\[
\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \frac{dx}{dt} \bigg|_{\text{traj}} \nabla_x + \frac{d\tilde{v}}{dt} \bigg|_{\text{traj}} \nabla_v \Rightarrow \frac{D}{Dt} N_s = 0
\]
Fluid interpretation – velocity and force is put inside the derivatives
\[
\frac{\partial}{\partial t} N_s + \nabla_x (\tilde{v} N_s) + \nabla_v \left[ \frac{q_s}{m_s} \left( \tilde{E}^m + \tilde{v} \times \tilde{B}^m \right) N_s \right] = 0
\]
analogy in the phase space to the continuity equation \( \partial_t \rho + \text{div}(\rho \tilde{v}) = 0 \)
Averaging over statistical ensemble \( f_s (\tilde{x}, \tilde{v}, t) \equiv \langle N_s (\tilde{x}, \tilde{v}, t) \rangle \)
Averaging of fields – one searches for average field that particle see
\[
\tilde{E} \equiv \langle \tilde{E}^m \rangle \quad \tilde{B} \equiv \langle \tilde{B}^m \rangle
\]
This field may be in general different from field in macroscopic Maxwell’s equations that is given by averaging over space (= problem of acting field – e.g. these fields differ in dielectrics).
In plasmas average acting field = Maxwell’s field!!

\[ N_s = f_s + \delta N_s \]
\[ \tilde{E}^m = \tilde{E} + \delta \tilde{E} \]
\[ \tilde{B}^m = \tilde{B} + \delta \tilde{B} \]

We split quantities to average values and deviations from them (fluctuations)

**Averaging of Klimontovich equation**

\[
\frac{\partial f_s (\tilde{x}, \tilde{v}, t)}{\partial t} + \tilde{v} \nabla_{\tilde{x}} f_s + \frac{q_s}{m_s} (\tilde{E} + \tilde{v} \times \tilde{B}) \nabla_{\tilde{v}} f_s = -\frac{q_s}{m_s} \left\langle \left( \delta \tilde{E} + \tilde{v} \times \delta \tilde{B} \right) \nabla_{\tilde{v}} \delta N_s \right\rangle
\]

left side - slowly varying quantities in space, time => collective effects
right side – correlations of fluctuations = collisions

It is basically generalized Boltzmann equations (additionally self-generated fields on the left side, on the right side not only binary correlations, but generalized collision term)
Averaging of Maxwell’s equations

\[
\langle \rho^m \rangle \equiv \rho = \sum_{e,i} q_s \int d\vec{v} \ f_s (\vec{x}, \vec{v}, t)
\]

\[
\langle \vec{J}^m \rangle \equiv \vec{J} = \sum_{e,i} q_s \int d\vec{v} \vec{v} \ f_s (\vec{x}, \vec{v}, t)
\]

averaging of charge and current densities after averaging ⇒ \textbf{macroscopic} Maxwell’s equations

\[
\text{div} \vec{E} = \frac{\rho}{\varepsilon_0} \quad \text{rot} \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0
\]

\[
\text{div} \vec{B} = 0 \quad \text{rot} \vec{B} = \mu_0 \vec{J} + \varepsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t}
\]

If all correlations are disregarded, collisionless kinetic equation with „self-generated fields“ is obtained

\[
\frac{\partial f_s}{\partial t} + \vec{v} \nabla f_s + \frac{q_s}{m_s} \left( \vec{E} + \vec{v} \times \vec{B} \right) \nabla_{\vec{v}} f_s = 0
\]  
\textbf{Vlasov equation}

Collisionless description for fast processes in ideal plasmas

Vlasov equation + Maxwell’s equations = closed system
We know from the Introduction
\[
\frac{\nu_c}{\omega_{pe}} \approx \frac{\ln \Lambda_e}{\Lambda_e} \approx \frac{\ln N_D}{N_D}
\]
in ideal plasmas \( N_D >> 1 \)

Hypothetical exercise – particle splitting
\( n_0 \to \infty \quad m_e \to 0 \quad \Rightarrow \omega_{pe} = \text{konst.} \quad e \to 0 \)

\( n_e e = \text{konst.} \quad e/m_e = \text{konst.} \quad T_e \to 0 \quad \Lambda_e \to \infty \quad \lambda_{De} = \text{konst.} \)

Size of fluctuations
\( \delta N_s \sim N_0^{1/2} \sim \Lambda_e^{1/2} \)

\( \delta E \sim e \delta N_s \sim \frac{1}{N_0} N_0^{1/2} = N_0^{-1/2} \sim \Lambda_e^{-1/2} \)
Kinetic equation (averaged Klimontovich equation)

Left side \( f_s \sim N_0 \sim \Lambda_e \)

Right side \( (\partial f / \partial t)_c = \delta E \delta N_s \approx \text{konst} \).

Right side can thus be disregarded for large \( \Lambda_e (N_D) \).

Krook collision term

Sometimes one needs to include collisions at least qualitatively.

For equilibrium (Maxwell) distribution \( f_M \), \( (\partial f / \partial t)_c = 0 \), any other distribution gradually approaches the equilibrium one via collisions

\[
(\partial f / \partial t)_c = -\nu_c (f - f_M)
\]

⇒ Krook collision term – the simplest one plausible

It can be generalized for mixtures.