Derivation of thermal energy conservation from kinetics

$$\langle \vec{v} \rangle = \vec{v}_s \equiv \vec{u} \qquad \vec{V} = \vec{v} - \vec{u}$$

$$f_s \equiv f \qquad m_s \equiv m \qquad \vec{F}_s = \vec{F} \qquad n_s \equiv n \qquad T_s \equiv T$$

Kinetic equation

$$\frac{\partial f}{\partial t} + \vec{v}\frac{\partial f}{\partial \vec{r}} + \frac{\vec{F}}{m}\frac{\partial f}{\partial \vec{v}} = \left(\frac{\partial f}{\partial t}\right)_c$$

We want to derive an equation for thermal energy

$$\int \frac{m}{2} V^2 d^3 V \qquad d^3 V = d^3 v$$

1st term

$$\int \frac{m}{2} (\vec{v} - \vec{u})^2 \frac{\partial f}{\partial t} d^3 v = \frac{\partial}{\partial t} \int \frac{m}{2} V^2 f \, d^3 v - \underbrace{\frac{m}{2} \frac{\partial \vec{u}}{\partial t} \int -2(\vec{v} - \vec{u}) f \, d^3 v}_{0} = \frac{\partial}{\partial t}\left(\frac{3}{2} n k_B T\right)$$

2nd term

$$\frac{m}{2} \int v_i \frac{\partial f}{\partial r_i} (v_j - u_j)^2 d^3 v = \frac{\partial}{\partial r_i} \int \frac{m v_i}{2} f (v_j - u_j)^2 d^3 v +$$

$$+ \frac{m}{2} 2 \frac{\partial u_j}{\partial r_i} \int d^3 v \, v_i (v_j - u_j) f =$$

$$= \frac{\partial}{\partial r_i} \int \frac{m}{2} (v_i - u_i)(\vec{v} - \vec{u})^2 f \, d^3 v + \frac{\partial}{\partial r_i} \int \frac{m u_i}{2} (\vec{v} - \vec{u})^2 f \, d^3 v +$$

$$+ \frac{\partial u_j}{\partial r_i} \int (v_i - u_i)(v_j - u_j) m f \, d^3 v + \frac{\partial u_j}{\partial r_i} u_i \int m (v_j - u_j) f \, d^3 v =$$

$$= \nabla \vec{q} + \nabla\left(\vec{u} \frac{3}{2} n k_B T\right) + P_{ij} \frac{\partial u_j}{\partial r_i} + 0$$

3rd term

$$\frac{m}{2}\int \frac{F_i}{m}\frac{\partial f}{\partial v_i}(\vec{v}-\vec{u})^2 d^3v = \underbrace{\int \frac{\partial}{\partial v_i}\left[\frac{F_i}{2} f (\vec{v}-\vec{u})^2\right] d^3v}_{0} -$$

$$- \frac{q}{2}\int \underbrace{\frac{\partial}{\partial v_i}\left[E_i + \varepsilon_{ijk} v_j B_k\right]}_{0}(\vec{v}-\vec{u})^2 d^3v - qE_i \underbrace{\int (v_i - u_i) f d^3v}_{0} -$$

$$- \frac{q}{2}\varepsilon_{ijk} B_k \int v_j \frac{\partial}{\partial v_i}(\vec{v}-\vec{u})^2 f d^3v =$$

$$= -\frac{q}{m}\varepsilon_{ijk} B_k \int m (v_j - u_j)(v_i - u_i) f d^3v - \frac{q}{m}\varepsilon_{ijk} B_k m \underbrace{\int u_j (v_i - u_i) f d^3v}_{0} =$$

$$= \underbrace{-\frac{q}{m}\varepsilon_{ijk} B_k P_{ij} = 0}_{P_{ij} = P_{ji} \quad \varepsilon_{ijk} = -\varepsilon_{jik}}$$

Volume force does not change the internal energy

Collisional term

$$\frac{1}{2}\int m v^2 \left(\frac{\partial f}{\partial t}\right)_c d^3v = \sum_t \int \frac{m_s v_s^2}{2}\left(\frac{\partial f}{\partial t}\right)_c^{(st)} dV_s \cong \sum_t \alpha_{ts}(T_t - T_s)$$

ss collisions – energy conservation → 0   st collisions – temperature relaxation

$$\frac{\partial}{\partial t}\left(\frac{3}{2} n_s k_B T_s\right) + \mathrm{div}\left(\vec{q}_s + \vec{V}_s \frac{3}{2} n_s k_B T_s\right) + P_{ij}\frac{\partial V_{si}}{\partial r_j} = \sum_t \alpha_{ts}(T_t - T_s)$$

for small temperature gradients ` $\vec{q}_s = -\kappa_s \nabla T_s$