

Lagrangeova pohybová rovnice

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0$$

Systém bez vazeb $\rightarrow q_i = x_i$ and $\dot{q}_i = v_i$

Lagrangeova funkce $L = T - V$ T – kinetická energie, V – potenciální energie

Elektrické pole je potenciální

$$L = \frac{mv^2}{2} - q\phi$$

Magnetické pole není potenciální

Místo polí potenciály

$$\vec{E}, \vec{B} \rightarrow \phi, \vec{A}$$

$$\vec{B} = \nabla \times \vec{A} = \text{curl } \vec{A} \rightarrow \text{div } \vec{B} = 0$$

$$\vec{E} = -\nabla\phi - \frac{\partial \vec{A}}{\partial t} \rightarrow \nabla \times \vec{E} = -\frac{\partial}{\partial t} \nabla \times \vec{A} = -\frac{\partial \vec{B}}{\partial t}$$

Symbol "curl" znamená "rot"

$$\vec{A}' = \vec{A} + \nabla\psi$$

Kalibrační podmínka

$$\phi' = \phi - \frac{\partial\psi}{\partial t}$$

Lorenzova kalibrace (Ludwig Lorenz)

Coulombova kalibrace

$$\text{div } \vec{A} + \frac{1}{c^2} \frac{\partial \phi}{\partial t} = 0$$

$$\Delta \phi - \frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2} = -\frac{\rho}{\epsilon_0}$$

$$\Delta \vec{A} - \frac{1}{c^2} \frac{\partial^2 \vec{A}}{\partial t^2} = -\mu_0 \vec{j}$$

$$\text{div } \vec{A} = 0$$

$$\Delta \phi = -\rho/\epsilon_0$$

$$\Delta \vec{A} - \frac{1}{c^2} \frac{\partial^2 \vec{A}}{\partial t^2} = \mu_0 \vec{j} +$$

$$+ \frac{1}{c^2} \nabla \frac{\partial \phi}{\partial t}$$

Zavedeme 4-vektory

$$A^\mu = \begin{pmatrix} \phi/c \\ \vec{A} \end{pmatrix} \quad \int_{\Sigma}^{\mu} \begin{pmatrix} e\mathcal{L} \\ \vec{j} \end{pmatrix} = \begin{pmatrix} q\vec{v} \cdot \vec{A} \\ q\vec{v} \cdot \vec{A} \end{pmatrix}$$

Hustota Lagrangianu

$$\mathcal{L} = -\int^{\mu} A_{\mu} = -q\phi \delta^3(\vec{x} - \vec{x}_0) + q(\vec{A} \cdot \vec{v}) \delta^3(\vec{x} - \vec{x}_0)$$

$$\tilde{\mathcal{L}} = \int \mathcal{L} d^3x = -q\phi + q\vec{A} \cdot \vec{v}$$

$$L = \frac{mv^2}{2} - q\phi + q\vec{A} \cdot \vec{v}$$

$$\frac{\partial L}{\partial v_i} = mv_i + qA_i \Rightarrow \frac{d}{dt} \frac{\partial L}{\partial v_i} = m\dot{v}_i + q\dot{A}_i + q(\vec{v} \cdot \nabla) A_i$$

$$\frac{\partial L}{\partial x_i} = -q \frac{\partial \phi}{\partial x_i} + qv_j \frac{\partial A_j}{\partial x_i}$$

⇒ Pohybová rovnice

$$m\dot{v}_i = \underbrace{-q \left(\frac{\partial A_i}{\partial t} + \frac{\partial \phi}{\partial x_i} \right)}_{-F_i} - \underbrace{qv_j \frac{\partial A_i}{\partial x_j} + qv_j \frac{\partial A_j}{\partial x_i}}_{q(\vec{v} \times \vec{B})_i}$$

Odvození magnetické síly (komponenta x)

$$\begin{aligned} F_x^{(B)} &= \cancel{qv_x \frac{\partial A_x}{\partial x}} - qv_y \frac{\partial A_y}{\partial x} + qv_z \frac{\partial A_z}{\partial x} - \cancel{qv_x \frac{\partial A_x}{\partial x}} - \\ &= -qv_y \frac{\partial A_x}{\partial y} - qv_z \frac{\partial A_x}{\partial z} = qv_y \underbrace{\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right)}_{B_z} - \\ &= -qv_z \underbrace{\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right)}_{B_y} = q(\vec{v} \times \vec{B})_x \end{aligned}$$

Hybnost částice

$$\vec{p}_0 = \frac{\partial L}{\partial \vec{v}} = m\vec{v} + q\vec{A}$$

$$\begin{aligned} \mathcal{E} &= \vec{v} \frac{\partial L}{\partial \vec{v}} - L = m\vec{v}^2 + q\vec{A}\vec{v} - \frac{1}{2}m\vec{v}^2 + q\phi - q\vec{A}\vec{v} \\ &= \frac{m\vec{v}^2}{2} + q\phi = \underbrace{\frac{(\vec{p} - q\vec{A})^2}{2m}} + q\phi = \mathcal{H} \end{aligned}$$

$$\dot{x} = \frac{\partial \mathcal{H}}{\partial \vec{p}} \quad \vec{p} = -\frac{\partial \mathcal{H}}{\partial \vec{x}} \quad \mathcal{H}$$

$$\dot{x} = \frac{\vec{p} - q\vec{A}}{m} = \vec{v}$$

$$\vec{p} = m\vec{v} + q\vec{A} = m \frac{\partial \vec{A}}{\partial \vec{x}} - q \frac{\partial \phi}{\partial \vec{x}}$$

$$\frac{dp_i}{dt} = q \frac{(p_i - qA_i)}{m} \frac{\partial A_i}{\partial x_i} - q \frac{\partial \phi}{\partial x_i}$$

$$m \frac{dv_i}{dt} + q \frac{\partial A_i}{\partial t} = q v_i \frac{\partial A_i}{\partial x_i} - q \frac{\partial \phi}{\partial x_i}$$

$$q \frac{\partial A_i}{\partial t} + q v_j \frac{\partial A_i}{\partial x_j}$$

$$\Rightarrow m \frac{dv_i}{dt} = q\vec{E} + q(\vec{v} \times \vec{B})_i$$

Relativistický popis

Polní část – relativistický invariant, kinetická část – modifikovat na invariantní

$$S = \int L dt \quad \text{— invariant}$$
$$\Rightarrow \int_p dt = \alpha \int \sqrt{1 - \frac{v^2}{c^2}} dt = \alpha \int \sqrt{1 - \frac{v^2}{c^2}} dt$$

Konstantu α lze najít z klasického $v \ll c$ limitu

$$L_p = \alpha \sqrt{1 - \frac{v^2}{c^2}} \approx \alpha \left(1 - \frac{v^2}{2c^2}\right) = \alpha - \alpha \frac{v^2}{2c^2}$$
$$-\alpha \frac{v^2}{2c^2} = \frac{mv^2}{2} \gg \alpha = mc^2$$

$$L = -mc^2 \sqrt{1 - \frac{v^2}{c^2}} - q\phi + q\vec{A}\vec{v}$$

$$\vec{p} = \frac{\partial L}{\partial \vec{v}} = \frac{m\vec{v}}{\sqrt{1 - \frac{v^2}{c^2}}} + q\vec{A}$$

$$E = \vec{v} \frac{\partial L}{\partial \vec{v}} - L = \frac{mc^2}{\sqrt{1 - \frac{v^2}{c^2}}} + q\phi$$

$$\frac{1}{c^2} (E - q\phi)^2 - (\vec{p} - q\vec{A})^2 = \frac{m^2(c^2 - v^2)}{1 - \frac{v^2}{c^2}}$$

$$E = c \sqrt{m^2 c^2 + (\vec{p} - q\vec{A})^2} + q\phi$$