One-fluid approximation (magnetohydrodynamics - MHD)

Mass conservation – electron and ion number conservation multiplied by masses and added

$$\frac{\partial n_e}{\partial t} + \mathrm{div}\left(n_e \vec{u}_e\right) = 0 \qquad\qquad \rho_M = m_e n_e + M_i n_i$$

$$\frac{\partial n_i}{\partial t} + \mathrm{div}\left(n_i \vec{u}_i\right) = 0 \qquad\qquad \vec{u} = \frac{m_e n_e \vec{u}_e + M_i n_i \vec{u}_i}{m_e n_e + M_i n_i}$$

$$\Rightarrow \quad \frac{\partial}{\partial t}\rho_M + \mathrm{div}\left(\rho_M \vec{u}\right) = 0 \qquad (1) \qquad\qquad \vec{u} \approx \vec{u}_i\,,\ \rho_M \simeq \rho_i$$

If we multiply electron and ion conservation by charges (q_e, q_i) and added up, we get charge conservation (equation for charge density ρ_c and current density $\vec{j}$)

$$\frac{\partial \rho_c}{\partial t} + \mathrm{div}\,\vec{j} = 0 \qquad (2)$$

Suggested reading: Chen 5.7, 6.2-6.4, 6.7, Nicholson chap. 8

Momentum conservation for electrons and ions may be expressed, as follows

$$\frac{\partial}{\partial t}\left(m_\alpha n_\alpha \vec{u}_\alpha\right)+\mathrm{div}\left(m_\alpha n_\alpha \vec{u}_\alpha \otimes \vec{u}_\alpha\right)=q_\alpha n_\alpha\left(\vec{E}+\vec{u}_\alpha\times\vec{B}\right)-\nabla p_\alpha-\sum_\beta m_\alpha n_\alpha \nu_{\alpha\beta}\left(\vec{u}_\alpha-\vec{u}_\beta\right)$$

where operator $\otimes$ is defined as follows $\quad \vec{a}\otimes\vec{b}=a_i\, b_j$

After summation one obtains

$$\frac{\partial}{\partial t}\left(\rho_M \vec{u}\right)+\mathrm{div}\left(\rho_M \vec{u}\otimes\vec{u}\right)=\rho_c\vec{E}+\vec{j}\times\vec{B}-\nabla p \qquad (3)$$

where $\quad \rho_c=q_e n_e+q_i n_i \qquad \vec{j}=q_e n_e \vec{u}_e+q_i n_i \vec{u}_i \qquad p=p_e+p_i$

(momentum exchange between electrons and ions $A_{ei}=-m_e n_e \nu_{ei}\left(\vec{u}_e-\vec{u}_i\right)$ a $A_{ie}=-M_i n_i \nu_{ie}\left(\vec{u}_i-\vec{u}_e\right)$ cancels)

To get equation for $\vec{j}$ it is possible to add momentum eqs multiplied $\left(\times q_\alpha / m_\alpha\right)$, but it will be difficult to close equation system without assumption $m_e \ll M_i$.

Let us start from assumption of negligible electron mass and let's require zero difference in electron and ion acceleration. While difference in acceleration due to gravitation is zero, ion acceleration by other forces is negligible compared to electrons and thus the total force acting on electrons is set to 0 (must be small)

$$0 = -\nabla p_e + q_e n_e \left(\vec{E} + \vec{u}_e \times \vec{B} \right) - m_e n_e \nu_{ei} \left(\vec{u}_e - \vec{u}_i \right)$$

Additionally, we use quasineutrality for slow motions and express electric field

$$\vec{E} = -\vec{u} \times \vec{B} - \frac{M_i}{e \rho_M Z} \nabla p_e + \frac{M_i}{e \rho_M Z} \vec{j} \times \vec{B} + \frac{m_e \nu_{ei}}{e^2 n_e} \vec{j} \qquad (4)$$

Current along magnetic field – electrons dominate

$$-\frac{e\vec{E}}{m_e} - \nu_{ei} \cdot \vec{u}_e \simeq 0 \quad \Rightarrow \quad \vec{j} = -e n_e \vec{u}_e = \frac{e^2 n_e}{m_e \nu_{ei}} \vec{E} \qquad \sigma_E = \frac{e^2 n_e}{m_e \nu_{ei}}$$ el. conductivity

and the last term of equation (4) may be transformed to the form $\dfrac{\vec{j}}{\sigma_E}$

From (4) we obtain equation for current

$$\vec{j}+\frac{M_i\sigma_E}{e\rho_M Z}\left(\vec{j}\times\vec{B}\right)=\sigma_E\left(\vec{E}+\vec{u}\times\vec{B}+\frac{M_i}{e\rho_M Z}\nabla p_e\right) \qquad (5)$$

to close the equation system, we express $p_e \simeq \alpha p$, where $\alpha = 1$ for $Z >> 1$ and $\alpha = 1/2$ for $Z = 1$, $T_e = T_i$

When Maxwell's equations and equation of state for pressure are added, one obtaines closed system of equations that can be solved

In MHD, equations are usually simplified by additional assumptions:

the Hall current is usually omitted compared to flow term $\vec{j}\times\vec{B} << \vec{u}\times\vec{B}$.

for low temperatures one omits pressure in the equation for current (pressure leads to Biermann battery term – *B* cannot arise form 0 in MHD without this term) then the current may be expressed

$$\Rightarrow \quad \vec{j}=\sigma_E\left(\vec{E}+\vec{u}\times\vec{B}\right) \quad \textit{\textbf{Ohm's law}} \qquad (6)$$

Ideal MHD $\left(\nu \to 0 \Rightarrow \sigma_E \to \infty\right)$

$$\frac{\partial \rho_M}{\partial t} + \mathrm{div}\left(\rho_M \vec{u}\right) = 0$$

$$\rho_M \frac{\mathrm{d}\vec{u}}{\mathrm{d}\vec{t}} = -\nabla p + \vec{j} \times \vec{B}$$

$$\mathrm{curl}\, \vec{B} = \mu_0 \vec{j}$$

$$\frac{\partial \vec{B}}{\partial t} = \mathrm{curl}\left(\vec{u} \times \vec{B}\right)$$

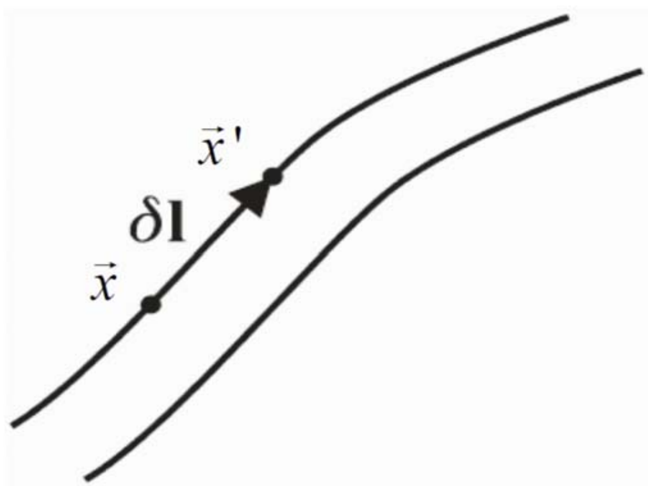
$$\vec{E} = -\vec{u} \times \vec{B}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\partial}{\partial t} + \vec{u}\nabla$$

Freezing of magnetic field into plasma

plasma moves along flux tubes, flux tube element is

$$\delta\vec{l} = \vec{x}' - \vec{x}$$



and velocity $\vec{u}'$ in point $\vec{x}'$ is $\vec{u}' = \vec{u} + \left(\delta\vec{l}\,\nabla\right)\vec{u}$, then time derivative of the element

$$\frac{\mathrm{d}}{\mathrm{d}\,t}\delta\vec{l} = \vec{u}' - \vec{u} = \left(\delta\vec{l}\,\nabla\right)\vec{u} \qquad .$$

The equation for time derivative of *B* is rearranged using well-known vector identity

$$\frac{\partial\vec{B}}{\partial t} = -\vec{B}\,\mathrm{div}\,\vec{u} + \left(\vec{B}\nabla\right)\vec{u} - \left(\vec{u}\nabla\right)\vec{B}$$

When this equation is combined with continuity relation, one obtains

$$\frac{\partial}{\partial t}\left(\frac{\vec{B}}{\rho}\right) + \left(\vec{u}\nabla\right)\left(\frac{\vec{B}}{\rho}\right) = \frac{\mathrm{d}}{\mathrm{d}\,t}\left(\frac{\vec{B}}{\rho}\right) = \left(\frac{\vec{B}}{\rho}\nabla\right)\vec{u}$$

The variations of vectors $\delta\vec{l}$ and $\vec{B}/\rho$ are given by the same equation, and thus magnetic force lines follow plasma motions, they are “***frozen***” into plasma.

For surface *S* moving together with plasma it holds $\frac{\mathrm{d}}{\mathrm{d}\,t}\int_S \vec{B}\,\mathrm{d}\vec{S} = 0$ (***Alfvén‘s theorem***)

Hydromagnetic equilibrium

We start from equations $\rho_M \frac{\mathrm{d}\vec{u}}{\mathrm{d}t} = -\nabla p + \vec{j} \times \vec{B}$ $\qquad$ $\mathrm{curl}\,\vec{B} = \mu_0 \vec{j}$

Equilibrium $\frac{\mathrm{d}\vec{u}}{\mathrm{d}t} \simeq \frac{\partial \vec{u}}{\partial t} = 0 \quad \Rightarrow \quad \nabla p = \vec{j} \times \vec{B}$

$$\vec{a} \times \left(\vec{b} \times \vec{c}\right) = \left(\vec{a} \cdot \vec{c}\right)\vec{b} - \left(\vec{a} \cdot \vec{b}\right)\vec{c} \quad \Rightarrow \quad \left(\vec{B} \times \nabla \vec{p}\right) = B^2 \vec{j} - \left(\vec{j} \cdot \vec{B}\right)\vec{B}$$

for component $\parallel B$ it always holds, $j_\parallel$ cannot be derived from here $\left(j_\parallel = \sigma_E E_\parallel\right)$

$$\vec{j}_\perp = \frac{\vec{B} \times \nabla \vec{p}}{B^2} \qquad \left(\mathrm{curl}\,\vec{B}\right) \times \vec{B} = \mu_0 \left(\vec{j} \times \vec{B}\right) = \mu_0 \left(\vec{j}_\perp \times \vec{B}\right) = \mu_0 \left(\frac{\vec{B} \times \nabla p}{B^2} \times \vec{B}\right) = \mu_0 \nabla p$$

$$\nabla \left(p + \frac{1}{\mu_0}\frac{B^2}{2}\right) = \underbrace{\frac{1}{\mu_0}\left(\vec{B}\nabla\right)\vec{B}}_{\text{often} = 0} \quad \Rightarrow \quad p + \frac{1}{\mu_0}\frac{B^2}{2} = \text{const.} \qquad \text{diamagnetic effect}$$

$B^2/2\mu_0$ = **magnetic pressure**

$$\beta = \frac{p}{\frac{B^2}{2\mu_0}} = \frac{\Sigma n_\alpha k_B T_\alpha}{\frac{B^2}{2\mu_0}}$$

ratio of thermal pressure to magnetic pressure

(parameter β of a device gives the ratio of maximal thermal pressure to the maximal magnetic pressure)

Non-ideal MHD – plasma diffusion into magnetic field

$$\nabla \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0 \qquad \vec{j} = \sigma_E \left(\vec{E} + \vec{u} \times \vec{B} \right)$$

we shall assume $u \simeq 0$

$$\text{curl } \vec{B} = \mu_0 \vec{j}$$

$$\frac{\partial \vec{B}}{\partial t} = -\frac{1}{\mu_0 \sigma_E} \nabla \times \left(\nabla \times \vec{B} \right) = \frac{1}{\mu_0 \sigma_E} \Delta \vec{B}$$

$$\tau = L^2 \mu_0 \sigma_E$$

τ is the time of plasma penetration into field

It is the $\vec{B}$ dissipation time – field energy transformation into heat

$$W = \frac{j^2}{\sigma_E} = \frac{1}{\sigma_E}\frac{B^2}{{\mu_0}^2 L^2}$$ where we have used $$\left|\vec{j}\right| = \frac{1}{\mu_0}\left|\text{curl}\,\vec{B}\right| = \frac{B}{\mu_0 L}$$

hence, the dissipated energy $$W\tau = \frac{1}{\sigma_E}\frac{B^2}{{\mu_0}^2 L^2}\cdot L^2\mu_0\sigma_E = \frac{B^2}{\mu_0}$$

Simultaneous flow and penetration (diffusion)

$$\frac{\partial\vec{B}}{\partial t} = \frac{1}{\mu_0\sigma_E}\Delta\vec{B} + \nabla\times\left(\vec{u}\times\vec{B}\right)$$

the first term is diffusion; the second term is freezing (field moves together with flow)

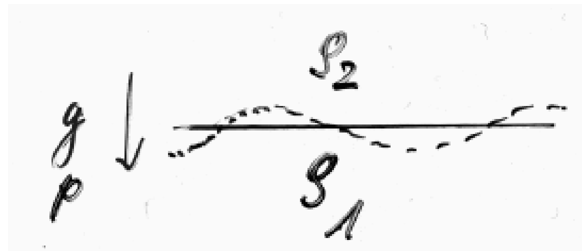
Magnetic Reynolds number

$$R_M = \frac{\textit{freezing term}}{\textit{diffusion term}} = \frac{\frac{1}{L}uB}{\frac{1}{\sigma_E\mu_0}\frac{1}{L^2}B} = \sigma_E\mu_0 uL$$

Instabilities driven by the pressure gradient

1.**Rayleigh-Taylor instability** boundary between fluids, if

$$\nabla p \cdot \nabla \rho < 0$$



Dispersion relation of waves

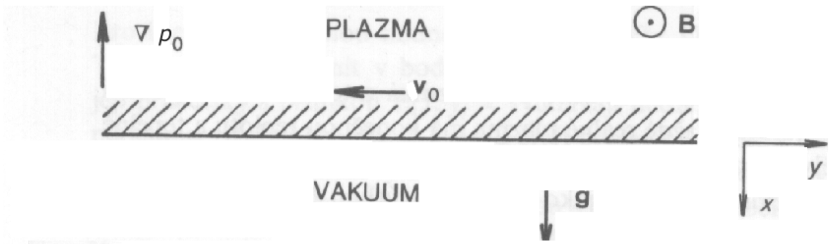
$$\omega^2 = \frac{kg\left(\rho_1 - \rho_2\right)}{\rho_1 + \rho_2}$$

for $\rho_2 < \rho_1$ waves on the fluid surface

for $\rho_2 > \rho_1$ $\omega_1 = i\gamma$ amplitude grows $\Rightarrow$ instability

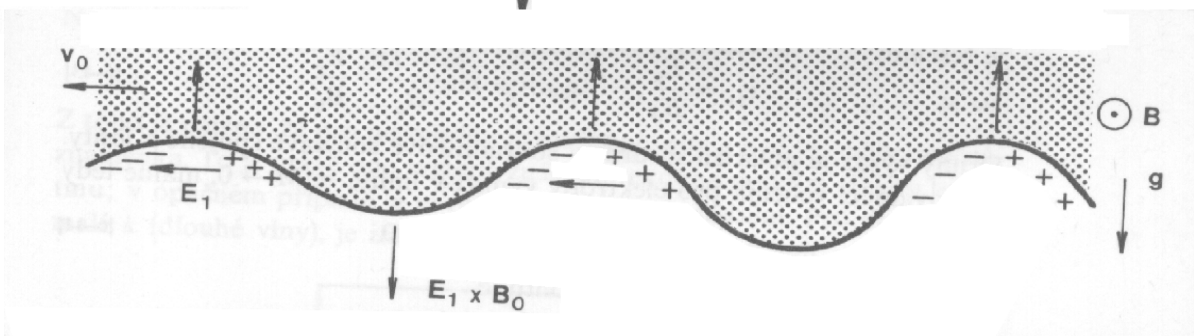
2.Instability of magnetically confined plasma (Kruskal-Schwartzschild)

B is the lighter fluid, plasma is the heavier fluid



Ion drift $\vec{v}_0 = \dfrac{m_i}{q_i}\dfrac{\vec{g}\times\vec{B}_0}{B_0^{\,2}}$,

electron drift may be omitted



Due to ion motion, charge is formed at the rippled surface, it induces electric field, and it causes E×B drift of ions and electrons that enhances ripples

Derivation from 2-fluid description (derivation is also possible from MHD):

ions – equation of motion

$$m_i\left[\frac{\partial\vec{v}_1}{\partial t}+\left(\vec{v}_0\nabla\right)\vec{v}_1\right]=q_i\left(\vec{E}_1+\vec{v}_1\times\vec{B}_0\right)$$

acceleration g is contained in velocity v_0

$$\vec{E}_1 = \left(0, E_y, 0\right) \qquad \vec{k} = \left(0, k_y, 0\right) \qquad -i\left(\omega - k_y \mathrm{v}_0\right)\vec{\mathrm{v}}_1 m_i = q_i\left(\vec{E}_1 + \vec{\mathrm{v}}_1 \times \vec{B}_0\right)$$

for $\left(\omega - k\mathrm{v}_0\right)^2 << \dfrac{q_i^{\,2} B_0^{\,2}}{m_i^{\,2}} = \Omega_c^{\,2} \Rightarrow \mathrm{v}_{1x} = \dfrac{E_{1y}}{B_0} \qquad \mathrm{v}_{1y} = -i\,\dfrac{\omega - k\mathrm{v}_0}{\Omega_c}\dfrac{E_{1y}}{B_0}$

v_{1x} is $E\times B$ drift (for ions and els), v_{1y} polarization drift (negligible for electrons)
ion continuity equation

$$\frac{\partial n_1}{\partial t} + \vec{\mathrm{v}}_0 \cdot \nabla n_1 + n_1 \operatorname{div} \vec{\mathrm{v}}_0 + \vec{\mathrm{v}}_1 \cdot \nabla n_0 + n_0 \operatorname{div} \vec{\mathrm{v}}_1 = 0$$

$$-i\omega n_1 + ik\mathrm{v}_0 n_1 + \mathrm{v}_{1x} \nabla n_0 + ik\mathrm{v}_{1y} n_0 = 0$$

electron continuity equation $-i\omega n_1 + \mathrm{v}_{1x}\nabla n_0 = 0$, where we assumed Z=1 and quasineutrality $n_{i1} = n_{e1}$ after substitution one obtains dispersion relation

$$\omega = \frac{1}{2}k\mathrm{v}_0 \pm \sqrt{\frac{1}{4}k^2 \mathrm{v}_0^{\,2} + \frac{\vec{g}\cdot\nabla n_0}{n_0}}$$

enough long waves grow, if the density gradient goes against the gravitational acceleration